$f(x) = x^3 + 3 \Rightarrow f'(x) = 3x^2 \qquad (f'(x))' = 6x$

3.8: Higher Derivatives

The derivative of a differentiable function f is also a function and it may have a derivative of its own:

$$(f')' = f'' \qquad \textbf{second derivative}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f'(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right) = \frac{d^2 f(x)}{dx^2}$$

`Alternative Notation:` If $y = f(x)$ then

$$y'' = f''(x) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D^2 f(x).$$

Similarly, the **third derivative** $f''' = (f'')'$ or

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = D^3 f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is denoted by $f^{(n)}(x)$:

$$y^{(n)} = f^{(n)}(x) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right)} = D^n f(x) = D\left(D^{n-1} f(x)\right)$$

EXAMPLE 1. *If* $y = x^5 + 3x + 1$ *find* $f^{(n)}(x)$

$y' = 5x^4 + 3$

$y'' = 20x^3$

$y''' = 60x^2$

$y^{(4)} = 120x$

$y^{(5)} = 120$

$y^{(6)} = 0$

$y^{(7)} = 0$

$y^{(8)} = 0 \ldots$

$\boxed{y^{(n)} = 0 \text{ for } n \geq 6}$

CONCLUSION: If $p(x)$ *is a polynomial of degree* n *then,* $p^{(k)}(x) = 0$ *for* $k \geq n + 1$.

EXAMPLE 2. *Find the second derivative of* $f(x) = \tan(x^3)$.

$$f'(x) = \sec^2(x^3)\cdot(x^3)' = 3x^2\sec^2(x^3)$$

$$f''(x) \overset{\text{P.R.}}{=} (3x^2)'\sec^2(x^3) + 3x^2\left(\underset{\downarrow\ \text{C.R.}}{\sec^2(x^3)}\right)'$$

$$= 6x\sec^2(x^3) + 3x^2\cdot 2\sec(x^3)\cdot\underbrace{(\sec(x^3))'}$$

$$= 6x\sec^2(x^3) + 6x^2\sec(x^3)\cdot\sec(x^3)\tan(x^3)\cdot 3x^2$$

Ex 3 $h(x)$ is twice differentiable

$$f(x) = \underbrace{h(\sqrt{x})}_{h\circ\sqrt{}} + \underbrace{\sqrt{h(x)}}_{\sqrt{}\circ h}$$

Find $f''(x)$

$$f'(x) = h'(\sqrt{x})\cdot(\sqrt{x})' + \frac{1}{2\sqrt{h(x)}}\cdot h'(x)$$

$$= \frac{h'(\sqrt{x})}{2\sqrt{x}} + \frac{h'(x)}{2\sqrt{h(x)}}$$

$$\frac{f'g-fg'}{g^2}$$

$$f''(x) = \frac{h''(\sqrt{x})(\sqrt{x})'\cdot 2\sqrt{x} - h'(\sqrt{x})\cdot(2\sqrt{x})'}{(2\sqrt{x})^2} + \frac{h''(x)\,2\sqrt{h(x)} - h'(x)\cdot(2\sqrt{h(x)})'}{(2\sqrt{h(x)})^2}$$

$$f''(x) = \frac{h''(\sqrt{x})\cdot\frac{1}{\cancel{2\sqrt{x}}}\cdot\cancel{2\sqrt{x}} - h'(\sqrt{x})\cdot\frac{\cancel{2}}{\cancel{2}\sqrt{x}}}{4x} + \frac{2h''(x)\sqrt{h(x)} - h'(x)\frac{\cancel{2}}{\cancel{2}\sqrt{h(x)}}h'(x)}{4h(x)}$$

EXAMPLE 4. *Find* $D^{2013}\sin x$.

$$\begin{array}{r} 503 \\ 4\,\overline{)\,2013} \\ \hline 1 \end{array}$$

$D\sin x = \cos x$

$D^2 \sin x = D(\cos x) = -\sin x$

$D^3 \sin x = D(-\sin x) = -\cos x$

$D^4 \sin x = D(-\cos x) = \sin x$

$D^5 \sin x = D(\sin x) = \cos x$

$503 \times 4 = 2012$

$D^{2012}(\sin x) = \sin x \quad \Rightarrow D^{2013}(\sin x) = D(D^{2012}\sin x)$

$= D(\sin x) = \boxed{\cos x}$

EXAMPLE 4. *If* $f(x) = \dfrac{1}{x}$ *find a general formula for its* n^{th} *derivative.*

$f(x) = \dfrac{1}{x^1}$

$f'(x) = -\dfrac{1}{x^2} = -x^{-2}$

$f''(x) = -(-2x^{-3}) = \dfrac{2}{x^3}$

$f'''(x) = -\dfrac{2 \cdot 3}{x^4}$

$f^{(n)} = (-1)^n \dfrac{1 \cdot 2 \cdot 3 \cdot \ldots n}{x^{n+1}}$

$f^{(4)}(x) = \dfrac{2 \cdot 3 \cdot 4}{x^5}$

Factorial

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$0! = 1$

$f^{(5)}(x) = -\dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{x^6}$

$$\boxed{f^{(n)} = \frac{(-1)^n \, n!}{x^{n+1}}}$$

Acceleration: If $s(t)$ is the position of an object then the acceleration of the object is the first derivative of the velocity (consequently, the acceleration is the second derivative of the position function.)

$$a(t) = v'(t) = s''(t).$$

$v(t) = s'(t)$

EXAMPLE 6. *If $s(t) = t^3 - \frac{9}{2}t^2 - 30t + 12$ is the position of a moving object at time t (where $s(t)$ is measured in feet and t is measured in seconds) find the acceleration at the times when the velocity is zero.*

$v(t) = s'(t) = 3t^2 - 9t - 30 = 3(t^2 - 3t - 10)$

$v(t) = 3(t-5)(t+2) = 0$

$t = 5$, $t = -2$ impossible

$a(t) = v'(t) = 6t - 9 \Rightarrow a(5) = 6 \cdot 5 - 9 = 21\ ft/s^2$

EXAMPLE 7. *Sketch the curve traced by* $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ *and plot the position, tangent and acceleration vectors at* $t = \dfrac{\pi}{4}$.

$\vec{r}\left(\frac{\pi}{4}\right) = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ position

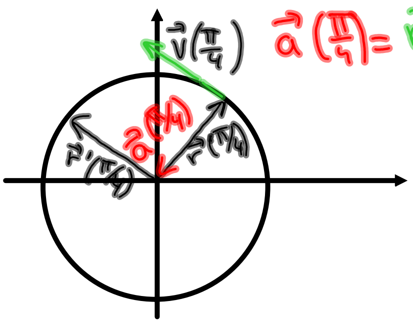
$\boxed{\vec{r}(t) \cdot \vec{r}\,'(t) = 0}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

tangent $\vec{r}\,'\left(\frac{\pi}{4}\right) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

acceleration $\vec{a}(t) = \vec{r}\,''(t) = \langle -\cos t, -\sin t \rangle = -\vec{s}(t)$

$\vec{a}\left(\frac{\pi}{4}\right) = \vec{r}\,''\left(\frac{\pi}{4}\right) = -\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$



Implicit second derivatives:

EXAMPLE 8. *Find* $y''(x)$ *if* $x^6 + y^6 = 66$.

$$x^6 + [y(x)]^6 = 66$$

Impl. diff.

$$6x^5 + 6y^5(x)\, y'(x) = 0 \Rightarrow y' = -\frac{x^5}{y^5}$$

Impl. diff. again

$$\cancel{30}^{5} x^4 + \cancel{30}^{5} y^4 (y')^2 + \cancel{6} y^5 y'' = 0$$

$$y^5 y'' = -5\left(x^4 + y^4 (y')^2\right)$$

$$y'' = -\frac{5\left(x^4 + y^4 (y')^2\right)}{y^5}$$

$$\boxed{y'' = -\frac{5\left(x^4 + y^4\left(-\frac{x^5}{y^5}\right)^2\right)}{y^5}}$$

Note that one can simplify it more!

$$y'' = -\frac{5}{y^5}\left(x^4 + y^4 \frac{x^{10}}{y^{10}}\right)$$

$$y'' = -\frac{5}{y^5} x^4 \left(1 + \frac{x^6}{y^6}\right) = -\frac{5x^4}{y^5} \cdot \frac{\overbrace{y^6 + x^6}^{=66}}{y^6}$$

$$y'' = -\frac{330x^4}{y^{11}}$$