4.1: Exponential functions and their derivatives

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. It is defined is the following manner:

x is integer

- If $x = n$, a positive integer, then $a^n = \underbrace{a \cdot a \cdot \quad \cdots \quad \cdot a}_{n \text{ factors}}$
- If $x = 0$ then $a^0 = 1$.
- If $x = -n$, n is a positive integer, then $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$

- If x is a rational number, $x = \frac{p}{q}$, with p and q integers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

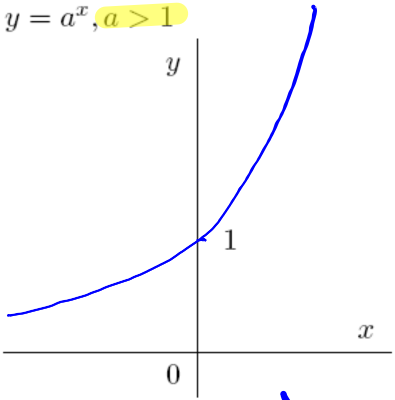
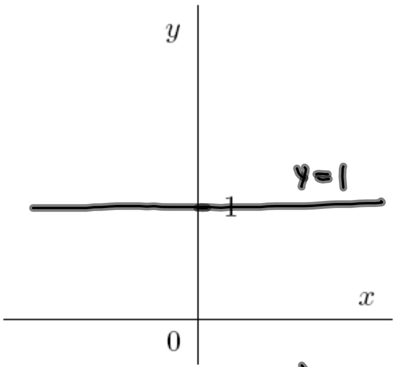
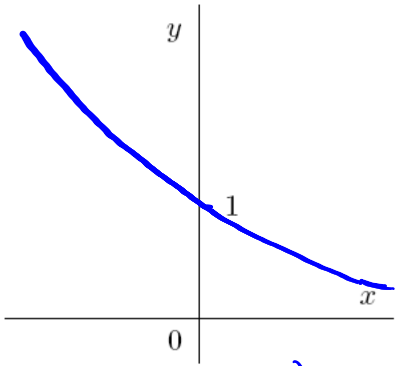
- If x is an irrational number then we define

$$a^x = \lim_{r \to x} a^r$$

where r is a rational number.

It can be shown that this definition uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

There are basically 3 kinds of exponential functions $y = a^x$:

Exponential growth	Constant	Exponential Decay
$y = a^x, a > 1$	$y = 1^x, a = 1$	$y = a^x, 0 < a < 1$
		
Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$
Range: $(0, \infty)$	Range: $\{1\}$	Range: $(0, \infty)$
$\lim\limits_{x\to\infty} a^x = \infty$		$\lim\limits_{x\to\infty} a^x = 0$
$\lim\limits_{x\to-\infty} a^x = 0$		$\lim\limits_{x\to-\infty} a^x = \infty$

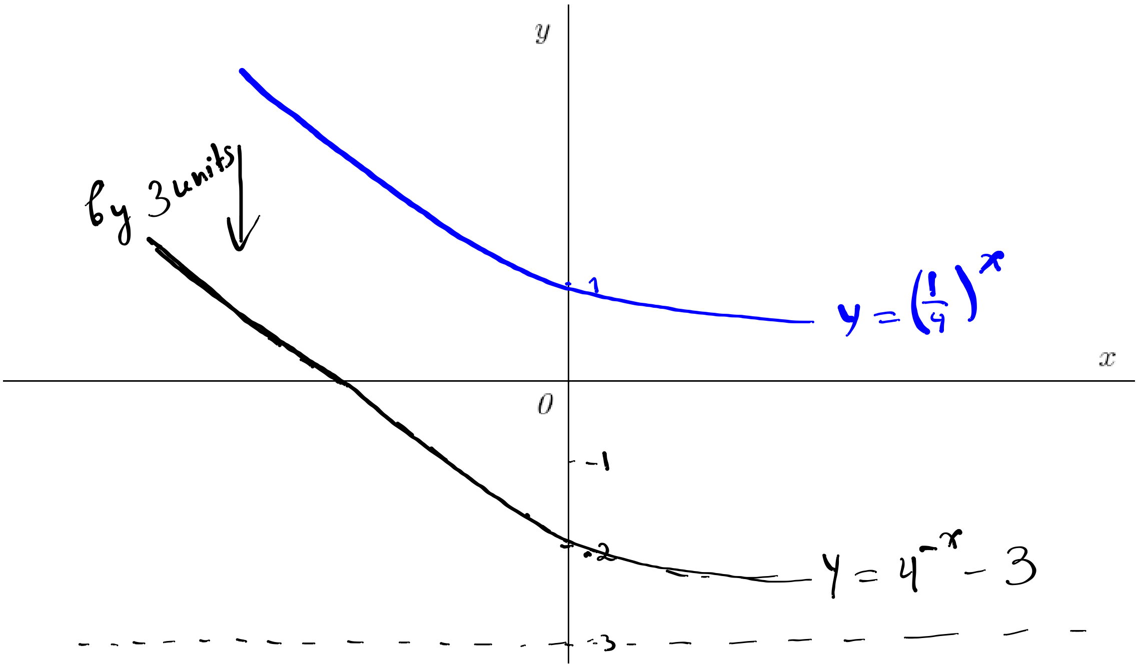
$y=0$ is horizontal asymptote

EXAMPLE 1. (a) *Find*

$$\lim_{x\to\infty}(4^{-x}-3) = \lim_{x\to\infty}\left(\frac{1}{4}\right)^x - 3 = 0-3 = \boxed{-3}$$

<u>Conclusion</u> $y=-3$ is horizontal asymptote

(b) *Sketch the graph of the function* $y = 4^{-x} - 3$ $=\left(\frac{1}{4}\right)^x-3$ *using transformations of graphs.*



EXAMPLE 2. *Find the limit:*

(a) $\lim_{x\to\infty}\left(\frac{\pi}{7}\right)^x = 0$

$0 < \frac{\pi}{7} < 1$

(b) $\lim_{x\to-\infty}(\pi^2-7)^x = 0$

$\pi^2 - 7 > 1$

(c) $\lim_{x\to 3+}\left(\frac{1}{7}\right)^{\frac{x}{x-3}} = \lim_{y\to\infty}\left(\frac{1}{7}\right)^y = 0$

$0 < \frac{1}{7} < 1$

$\boxed{y = \frac{x}{x-3} \xrightarrow[x\to 3^+]{} \infty \quad x > 3}$

There are in fact a variety of ways to define e. Here are a two of them:

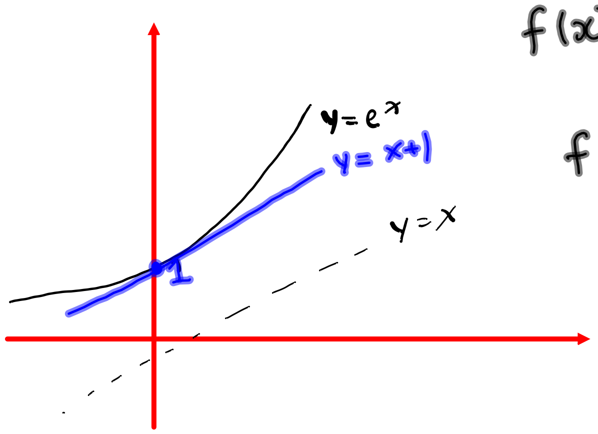
$a = e > 1$

1. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

2. e is the unique positive number for which $\boxed{\lim_{h\to 0}\frac{e^h-1}{h}=1.}$

It can be also shown that $\boxed{e \approx 2.71828.}$

Geometric interpretation:

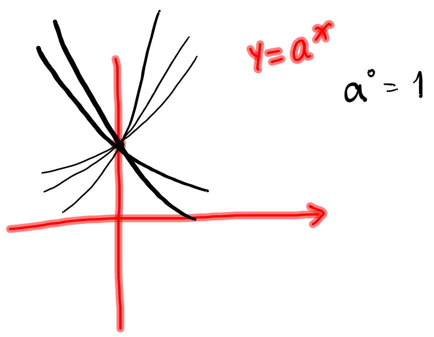


$f(x) = e^x$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{e^h - e^0}{h}$$

$$= \lim_{h\to 0}\frac{e^h-1}{h} = 1$$



Tangento to $y=e^x$ at $x=0$

$f(0) = e^0 = 1$

$y - 1 = 1(x-0)$

$y = x+1$

$e > 1$

EXAMPLE 3. *Find the limit:*

(a) $\lim_{x \to 1^+} e^{\frac{4}{x-1}} = \lim_{y \to \infty} e^y = \infty$

$y = \frac{4}{x-1} \xrightarrow[x \to 1^+]{} \infty$

$x > 1$

(b) $\lim_{x \to 1^-} e^{\frac{4}{x-1}} = \lim_{y \to -\infty} e^y = 0$

$y = \frac{4}{x-1} \xrightarrow[x \to 1^-]{} -\infty$

$x < 1$

(c) $\lim_{x \to \infty} \frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}}$

Way 1 $\quad e^{-5x} = \frac{1}{e^{5x}}$ → common denom.

Way 2 $\quad$ Multiply num. and denom. by e^{5x}

Way 3 $\quad$ Divide — " ——— " ———

Way 4 $\quad$ Replace $y = e^{5x} \Rightarrow e^{-5x} = \frac{1}{y}$

$$y = e^{5x} \xrightarrow[x \to \infty]{} \infty$$

$$\lim_{x \to \infty} \frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}} = \lim_{y \to \infty} \frac{y - \frac{1}{y}}{y + \frac{1}{y}} =$$

$$= \lim_{y \to \infty} \frac{\frac{y^2-1}{y}}{\frac{y^2+1}{y}} = \lim_{y \to \infty} \frac{y^2-1}{y^2+1} = \frac{1}{1} = \boxed{1}$$

Derivative of exponential function.

EXAMPLE 4. *Find the derivative of $f(x) = e^x$.*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} =$$

$$= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{=1} = e^x \cdot 1 = e^x$$

CONCLUSIONS:

$$(e^x)' = e^x$$

1. e^x is differentiable function.

2. If $u(x)$ is a differentiable function then by Chain Rule: $\boxed{\frac{\mathrm{d}}{\mathrm{d}x} e^{u(x)} = e^u \frac{\mathrm{d}u}{\mathrm{d}x}.}$

EXAMPLE 5. *Find y'' for e^{x^2}.*

$$y' = \left(e^{x^2}\right)' = e^{x^2}\left(x^2\right)' = 2xe^{x^2}$$

$$y'' = \left(2xe^{x^2}\right)' = 2\left[1\cdot e^{x^2} + x\cdot\left(e^{x^2}\right)'\right]$$

$$= 2\left[e^{x^2} + x\cdot 2x\, e^{x^2}\right]$$

$$= 2e^{x^2}\left[1 + 2x^2\right]$$

EXAMPLE 6. *Find the derivative:*

(a) $y = \sqrt{e^x + x^3}$

$u = e^x + x^3$

$$y' = \left(\sqrt{u(x)}\right)' = \frac{1}{2\sqrt{u}} \cdot u' = \frac{1}{2\sqrt{e^x + x^3}} \left(e^x + 3x^2\right)$$

(b) $y = e^{x \sin x}$

$u = x \sin x$

$$y' = (e^u)' = e^u \cdot u' = e^{x \sin x} \left(\sin x + x \cos x\right)$$

EXAMPLE 7. *For what value(s) of A does the function $y = e^{Ax}$ satisfy the equation $y'' + 2y' - 8y = 0$?*

$$\begin{cases} y = e^{Ax} \\ y' = Ae^{Ax} \\ y'' = (Ae^{Ax})' = A \cdot A e^{Ax} = A^2 e^{Ax} \end{cases}$$

<u>Note</u> $D^n(e^{Ax}) = A^n e^{Ax}$

$$y'' + 2y' - 8y = 0$$

$$A^2 e^{Ax} + 2Ae^{Ax} - 8e^{Ax} = 0$$

$$e^{Ax}(A^2 + 2A - 8) = 0$$

$$\underbrace{e^{Ax}}_{\neq 0}(A+4)(A-2) = 0 \implies \boxed{A = -4 \quad \text{OR} \quad A = 2}$$

<u>Conclusion</u> The functions $y = e^{-4x}$ and $y = e^{2x}$ are solutions of the given differential equation.