4.2:Inverse Functions

DEFINITION 1. *A function of domain X is said to be a* **one-to-one** *function if no two elements of X have the same image, i.e.*

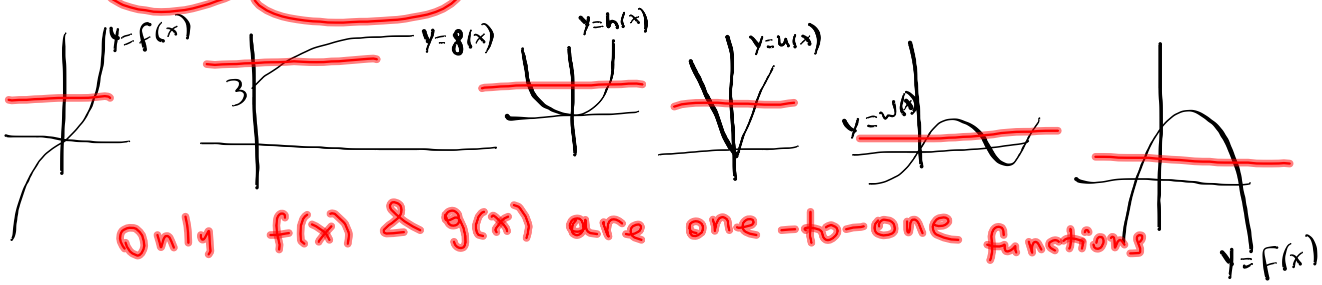
if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Horizontal line test:A function if one-to-one is and only if no horizontal line intersects its graph more once.

EXAMPLE 2. *Are the following functions one-to-one?*

$$f(x) = x^3, \quad g(x) = \sqrt{x} + 3, \quad h(x) = x^2, \quad u(x) = |x|, \quad w(x) = \sin x, \quad F(x) = -x^2 + x + 1$$



Only f(x) & g(x) are one-to-one functions

EXAMPLE 3. *Prove that* $f(x)=\dfrac{x-3}{x+3}$ *is one-to-one.*

If $f(x_1)=f(x_2) \overset{?}{\Longrightarrow} x_1=x_2$

$$\frac{x_1-3}{x_1+3}=\frac{x_2-3}{x_2+3}$$

$$(x_1-3)(x_2+3)=(x_2-3)(x_1+3)$$

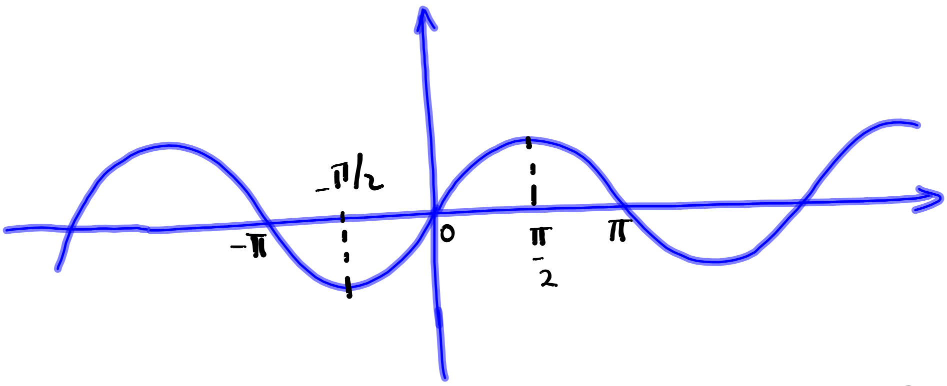
$$\cancel{x_1x_2}-3x_2+3x_1-\cancel{9}=\cancel{x_2x_1}-3x_1+3x_2-\cancel{9}$$

$$-3(x_2-x_1)=3(x_2-x_1)$$

$$6(x_2-x_1)=0$$

$$x_2-x_1=0 \Rightarrow \boxed{x_1=x_2}$$

EXAMPLE 4. *How we can restrict the domain of $f(x) = \sin x$ to make it one-to-one?*



For example, $D(\sin x) = [-\frac{\pi}{2}, \frac{\pi}{2}]$

DEFINITION 5. *Let f be a one-to-one function with domain X and range Y. Then the inverse function f^{-1} has the domain Y and range X and is defined for any y in Y by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y.$$

REMARK 6. Reversing roles of x and y in the last formula we get:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

REMARK 7. If $y = f(x)$ is one-to-one function with the domain X and the range Y then

$f \circ f^{-1} = f^{-1} \circ f$

for every x in X $\quad f^{-1}(f(x)) = f^{-1}(y) = x$

and

for every x in Y $\quad f(f^{-1}(x)) = f(y) = x$

$\Rightarrow$

CAUTION: $f^{-1}(x)$ **does NOT mean** $\dfrac{1}{f(x)}$.

TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f:

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 8. *(cf. Example3) Find the inverse function of* $f(x) = \dfrac{x-3}{x+3}$.

1 $$y = \frac{x-3}{x+3}$$

2 $$y(x+3) = x-3$$

$$xy + 3y = x - 3 \Rightarrow xy - x = -3y - 3$$

$$x(y-1) = -3(y+1)$$

$$x = -\frac{3(y+1)}{y-1}$$

3 $$y = -\frac{3(x+1)}{x-1}$$

$$\boxed{f^{-1}(x) = -\frac{3(x+1)}{x-1}}$$

Note

	Domain	Range
f	$x \neq -3$	$y \neq 1$
f^{-1}	$x \neq 1$	$y \neq -3$

Because domain f = range f^{-1}

EXAMPLE 9. *Given* $f(x) = x^2 + x$, $\underbrace{x \geq \frac{1}{2}}_{\text{domain}}$. *Find the inverse function of* f.

so that $f(x)$ is one to one

$y = x^2 + x$

$x^2 + x - y = 0$

Recall $ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case: $a = 1$, $b = 1$, $c = -y$

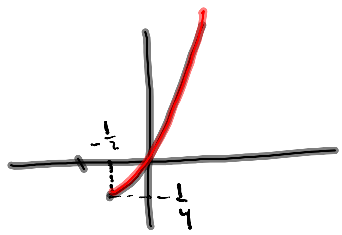
$$x_{1,2} = \frac{-1 \pm \sqrt{1+4y}}{2} \Rightarrow x = \frac{-1 + \sqrt{1+4y}}{2}$$

$$y = \frac{-1 + \sqrt{1+4x}}{2}$$

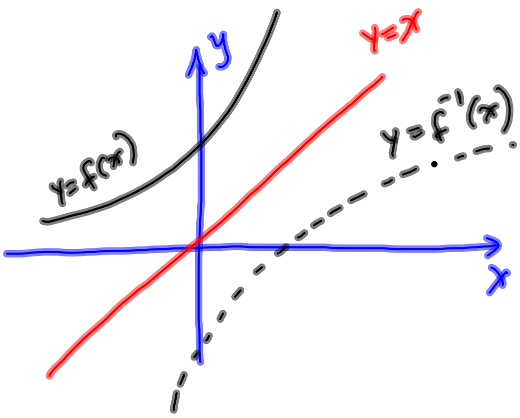
$$f^{-1}(x) = \frac{-1 + \sqrt{1+4x}}{2}$$

Note if we consider $y = x^2 + x$, $x \geq -\frac{1}{2}$ then

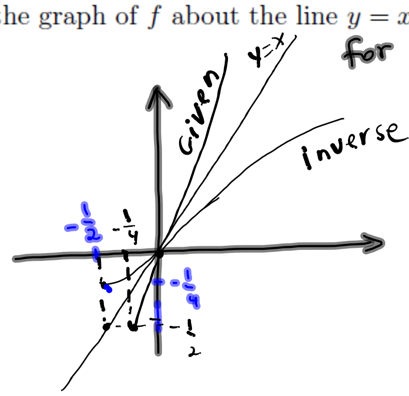
	domain	Range
f	$x \geq -\frac{1}{2}$	$y \geq -\frac{1}{4}$
f^{-1}	$x \geq -\frac{1}{4}$	$y \geq -\frac{1}{2}$



FACT: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.



for Ex. 9



Remark If (a, b) belongs to the graph of $y = f(x)$ then (b, a) belongs to the graph of $y = f^{-1}(x)$

THEOREM 10. *If f is a one-to-one differentiable function with $\underbrace{\text{inverse function } g = f^{-1}}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and*

$$\boxed{g'(a) = \frac{1}{f'(g(a))}.}$$

$$\Downarrow$$

$$f^{-1}(x) = g(x)$$

Proof.

$$f(f^{-1}(x)) = x$$

$$f(g(x)) = x \overset{\text{Chain Rule}}{\Longrightarrow} f'(g(x))\, g'(x) = 1$$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))} \quad \Rightarrow \quad g'(a) = \frac{1}{f'(g(a))}$$

EXAMPLE 11. *Suppose that g is the inverse function of f and $f(4) = 5$, $f'(4) = 7$. Find $g'(5)$.*

$$g(x) = f^{-1}(x)$$

$$\Downarrow$$

$$f^{-1}(f(4)) = f^{-1}(5)$$

$$4 = f^{-1}(5)$$

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(4)} = \boxed{\frac{1}{7}}$$

$$g(5) = f^{-1}(5) = 4$$

EXAMPLE 12. *Suppose that g is inverse of f. Find $g'(a)$ where*

$$g'(a) = \frac{1}{f'(g(a))}$$

(a) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$

$$g'(2) = \frac{1}{f'(g(2))}$$

We don't know how to find formula for inverse. To determine $g(2)$ we have to guess for what x $f(x) = 2$ (b/c then $x = f^{-1}(2) = g(2)$)

Guess

$f(1) = \sqrt{1+1+1+1} = 2$

$\Rightarrow g(2) = 1$

Then

$$g'(2) = \frac{1}{f'(1)}$$

Find

$$f'(x) = \frac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$$

$$f'(1) = \frac{3+2+1}{2 \cdot 2} = \frac{6}{4} = \frac{3}{2}$$

$$g'(2) = \frac{1}{3/2} = \boxed{\frac{2}{3}}$$

(b) $f(x) = \dfrac{2x-3}{x+3}$, $a = \dfrac{1}{2}$.

$$g'\left(\tfrac{1}{2}\right) = \frac{1}{f'\left(g\left(\tfrac{1}{2}\right)\right)}$$

Find x such that $f(x) = \frac{1}{2}$ (b/c then $x = f^{-1}\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right)$)

$$\frac{1}{2} = \frac{2x-3}{x+3}$$

$x + 3 = 4x - 6$

$\boxed{x = 3}$

$f(3) = \frac{1}{2} \Rightarrow g\left(\frac{1}{2}\right) = 3$

$f'(3) = \frac{1}{4}$ (Check it!)

Finally, $g'\left(\frac{1}{2}\right) = \dfrac{1}{f'(3)} = \dfrac{1}{1/4} = \boxed{4}$

$f^{-1}(x) = g(x)$

$$g'(a) = \frac{1}{f'(g(a))}$$

$f(b) = a$

$\Downarrow$

$b = f^{-1}(a) = g(a)$

(c) $f(x) = 4 + 3x + e^{3(x-1)}$, $a = 8$.

Find x such that

$f(x) = 8$

guess

$x = 1 \Rightarrow f(1) = 4 + 3 + 1 = 8$

$f(1) = 8 \Rightarrow 1 = f^{-1}(8) = g(8)$

$$g'(8) = \frac{1}{f'(g(8))} = \frac{1}{f'(1)} = \frac{1}{0 + 3 + 3e^{3(x-1)}}\Bigg|_{x=1}$$

$$g'(8) = \boxed{\frac{1}{6}}$$