4.3:Logarithmic Functions = inverse of the exponential function.

DEFINITION 1. *The exponential function* $f(x) = a^x$ *with* $a \neq 1$ *is a one-to-one function. The inverse of this function, called the* **logarithmic function** *with* **base** a, *is denoted by* $f^{-1}(x) = \log_a x$.

Namely,

a is base

$$\boxed{\log_a x = y \quad \Leftrightarrow \quad a^y = x.}$$

$x > 0$

In other words, if $x > 0$ then $\log_a(x)$ is the exponent to which the base a must be raised to give x.

EXAMPLE 2. *Evaluate*

(a) $\log_2 16 = \log_2 2^4 = 4$

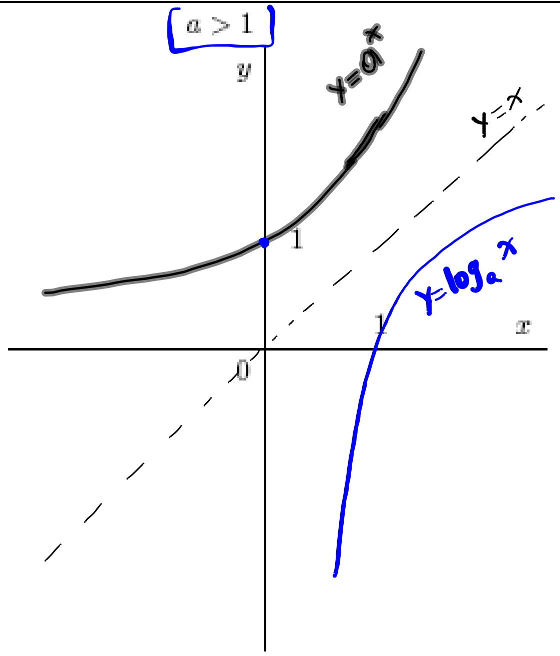
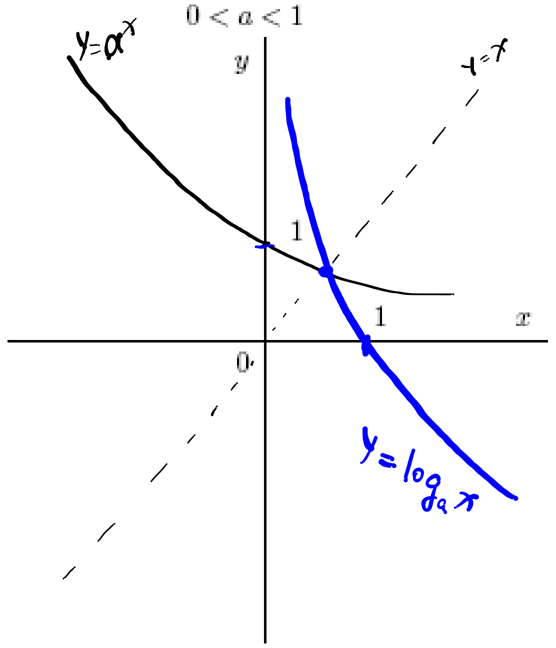
(b) $\log_3 \frac{1}{81} = \log_3 3^{-4} = -4$

(c) $\log_{125} 5 = \log_{5^3} 5 = \frac{1}{3}$ because $(5^3)^{\frac{1}{3}} = 5$

CANCELLATION RULES:

- $\log_a a^x = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of logarithmic functions $y = \log_a x$:

$\boxed{a > 1}$	$0 < a < 1$
	
Domain: $x>0$	Domain: $x>0$
Range: $\mathbb{R}$	Range: $\mathbb{R}$
$\lim\limits_{x\to\infty} \log_a x = \infty$	$\lim\limits_{x\to\infty} \log_a x = \infty$
$\lim\limits_{x\to 0^+} \log_a x = -\infty$	$\lim\limits_{x\to 0^+} \log_a x = -\infty$
Vertical asymptote: $x=0$	Vertical asymptote: $x=0$
Horizontal asymptote: no	Horizontal asymptote: NO

Properties: Assume that $a \neq 1$ and $x, y > 0$.

$$\log_a(xy) = \log_a x + \log_a y$$

$$\boxed{\begin{gathered}\log_a\left(\frac{x}{y}\right) = \log_a x - \log_b y \\ \log_a(x^y) = y \log_a x\end{gathered}}$$

Notation: *Common Logarithm:* $\log x = \log_{10} x$. (Thus, $\log x = y \quad \Leftrightarrow \quad 10^y = x$.)
Natural Logarithm: $\ln(x) = \log_e(x)$. (Thus $\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$)

Proof

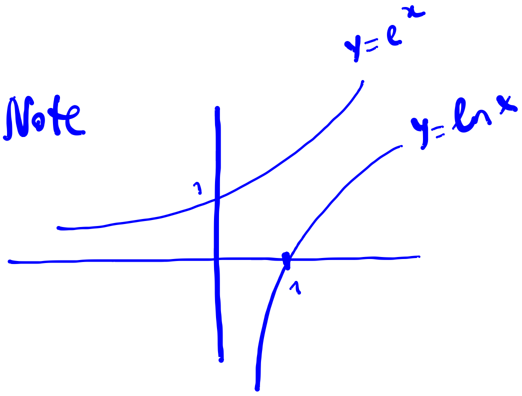
$$a^{\log_a(xy)} = a^{\log_a x + \log_a y} \qquad a^{m+n} = a^m \cdot a^n$$

Cancellation rule

$$xy = a^{\log_a x} \cdot a^{\log_a y}$$

$$xy = x \cdot y$$

Note



, because $e > 1$

Properties of the natural logarithms:

- $\ln(e^x) = x \ln e = x$
- $e^{\ln x} = e^{\log_e x} = x$

} because $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

- $\ln e = \log_e e = 1$
- $\log_a x = \dfrac{\ln x}{\ln a}$, where $a > 0$ and $a \neq 1$; $\Rightarrow$ Proof: $\ln a \log_a x = \ln x$
- $\lim_{x\to\infty} \ln x = \infty$
- $\lim_{x\to 0+} \ln x = -\infty$

$$e^{\ln a \cdot \log_a x} = e^{\ln x}$$

$$e^{mn} = (e^m)^n$$

$$\left(e^{\ln a}\right)^{\log_a x} = x$$

$$a^{\log_a x} = x$$

$$x = x$$

Cancellation rule

EXAMPLE 3. *Find each limit:*

(a) $\lim_{x\to\infty} \ln(x^2 - x) =$ $\lim_{y\to\infty} \ln y = \infty$

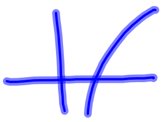
$y = x^2 - x$

$\lim_{x\to\infty} y = \infty$

(b) $\lim_{x\to 13+} \log_{13}(x - 13) =$ $\lim_{y\to 0^+} \log_{13} y = -\infty$

$13 > 1$

$y = x - 13 \xrightarrow[x\to 13^+]{} 0^+$



(c) $\lim_{x\to 0+} \log(\sin x) =$ $\lim_{y\to 0^+} \log(y) = \lim_{y\to 0^+} \log_{10} y = -\infty$

$y = \sin x \xrightarrow[x\to 0^+]{} 0^+$

$10 > 1$

(d)) $\lim_{x\to1}\underbrace{(\ln x)^{\sin x}}_{\substack{\text{continuous}\\ \text{at } x=1}} = (\ln 1)^{\sin 1} = 0^{\sin 1} = 0.$

Thus, use direct substitution rule

EXAMPLE 4. *Find the domain of* $f(x) = \ln(x^3 - x)$.

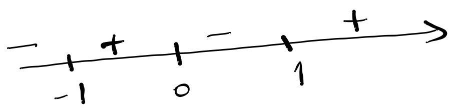
Domain of $y = \ln x$ is $x > 0$

Thus domain of $f(x)$ is

$$x^3 - x > 0$$

$$x(x^2 - 1) > 0$$

$$x(x-1)(x+1) > 0$$



$D(f) = (-1, 0) \cup (1, \infty)$

EXAMPLE 5. *Solve the following equations:*

(a) $\log_{0.5}(\overbrace{\log(x+120)}^{\star}) = -1$

You have to

OR

find first domain and then verify that your solutions are in that domain.

Solve equation and then plug in your solutions to the given equation

$$\log_{0.5}(\star) = -1 \Rightarrow \star = 0.5^{-1} = 2$$

$$\log_{10}(x+120) = 2$$

$$10^2 = x+120$$

$$100 = x+120$$

$$\Downarrow$$

$$\boxed{x=-20}$$

Verify that $x=-20$ is in domain

(b) $e^{5+2x} = 4$

$$\boxed{\ln e^{a} = a}$$

$$\ln e^{5+2x} = \ln 4$$

$$5 + 2x = \ln 4$$

$$2x = \ln 4 - 5$$

$$x = \boxed{\frac{\ln 4 - 5}{2}}$$

Note $\ln 4 = \ln 2^2 = 2\ln 2$

(c) $\log(x-1) + \log(x+1) = \log 15$

$$\log(x-1)(x+1) = \log 15$$

$$\log(x^2-1) = \log 15$$

$$x^2 - 1 = 15$$

$$x^2 = 16$$

$$x = \pm 4$$

because $\log(x-1)$ is undefined for $x = -4$

$$\boxed{x = 4}$$

(d) $\ln x^2 - 2\ln\sqrt{x^2+1} = 1$

$$\ln x^2 - \ln\left(\sqrt{x^2+1}\right)^2 = 1$$

$$\ln x^2 - \ln(x^2+1) = 1$$

$$\ln \frac{x^2}{x^2+1} = 1 \qquad \ln a = b \Rightarrow a = e^b$$

$$\frac{x^2}{x^2+1} = e^1$$

$$x^2 = e(x^2+1) \Rightarrow x^2 = e x^2 + e$$

$$x^2(1-e) = e$$

$$x^2 = \frac{e}{1-e} < 0$$

$$\boxed{\text{no solution}}$$

$$1-e \approx 1-2.7 < 0$$

EXAMPLE 6. *Find the inverse of the function:*

(a) $f(x) = \ln(x + 12)$

$$\boxed{a = \ln(b) \Leftrightarrow e^a = b}$$

$$y = \ln(x+12)$$

$$x + 12 = e^y$$

$$x = e^y - 12 \xRightarrow{x \Leftrightarrow y} y = e^x - 12$$

$$\boxed{f^{-1}(x) = e^x - 12}$$

(b) $f(x) = \dfrac{10^x - 1}{10^x + 1}$ $\Rightarrow$ $y = \dfrac{10^x - 1}{10^x + 1}$

$$(10^x + 1)y = 10^x - 1$$

$$10^x y + y = 10^x - 1$$

$$10^x y - 10^x = -1 - y$$

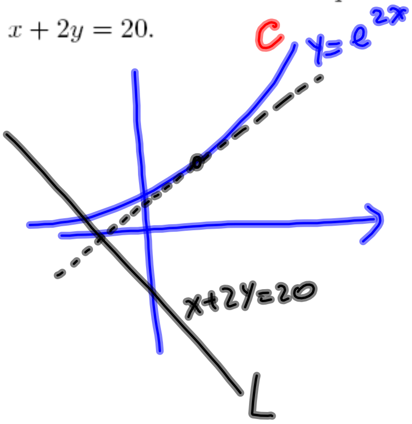
$$10^x (y - 1) = -(1 + y)$$

$$10^x = -\frac{1+y}{y-1}$$

$$10^x = \frac{1+y}{1-y} \Rightarrow x = \log_{10} \frac{1+y}{1-y}$$

$$y = \log \frac{1+x}{1-x} \Rightarrow \boxed{f^{-1}(x) = \log \frac{1+x}{1-x}}$$

EXAMPLE 7. *Find an equation of the tangent to the curve* $y = e^{2x}$ *that is perpendicular to the line* $x + 2y = 20$.



$m_L = -\frac{1}{2}$

m_c = slope of the tangent

$C \perp L$

$\Downarrow$

$m_L \cdot m_c = -1$

$$m_c = -\frac{1}{m_L} = -\frac{1}{-\frac{1}{2}} = 2$$

On the other hand $m_c = (e^{2x})' = 2e^{2x}$

$$\Rightarrow 2e^{2x} = 2$$

$$e^{2x} = 1$$

$$2x = 0$$

$$\Downarrow$$

Tangent point $\rightarrow$ $x = 0$

$$y = e^{2x}\Big|_{x=0} = 1$$

$$y - 1 = 2(x - 0)$$

$$\boxed{y = 2x + 1}$$

Change of Base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

In particular,

$$\boxed{\log_a x = \frac{\ln x}{\ln a}.} \quad , b = e$$

EXAMPLE 8. *Using calculator evaluate* $\log_2 15$ *to 4 decimal places.*

$$\log_2 15 = \frac{\ln 15}{\ln 2} \approx \frac{2.7081}{0.6931} \approx 3.9069$$