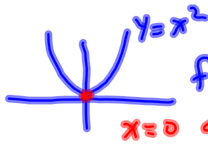
5.2: Maximum and Minimum Values



$f(0) \le f(x)=x^2$

$x=0$ abs. min.

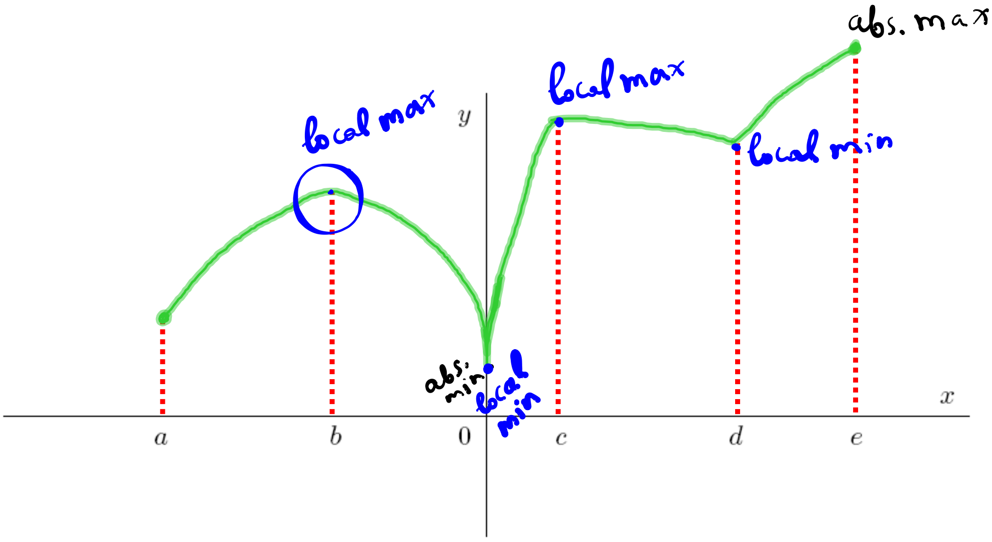
DEFINITION 1. *Let D be the domain of a function f.*

- *A function f has an* **absolute maximum** *(or* **global maximum***) at $x=c$ if $f(c) \geq f(x)$ for all x in D. In this case, we call $f(c)$ the* **maximum value.**
- *A function f has an* **absolute minimum** *(or* **global minimum***) at $x=c$ if $f(c) \leq f(x)$ for all x in D. In this case, we call $f(c)$ the* **minimum value.**

  The maximum and minimum values of f on D are called the **extreme values** *of f.*

relative

DEFINITION 2. *A function f has a local maximum at $x=c$ if $f(c) \geq f(x)$ when x is near c (i.e. in a neighborhood of c). A function f has a local minimum at $x=c$ if $f(c) \leq f(x)$ when x is near c.*

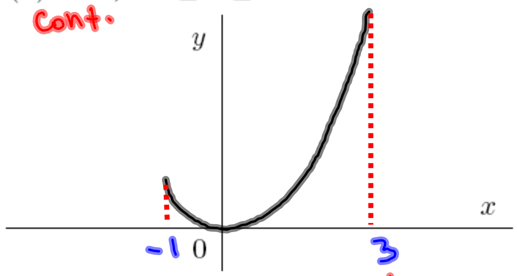


EXAMPLE 3. *Find the absolute and local extrema of f by sketching its graph:*

closed

(a) $f(x) = x^2$, $-1 \leq x \leq 3$

Cont.

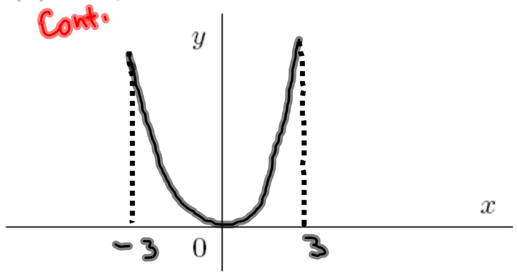


abs. extemum

	Local	*Absolute*	*Value*
Maximum	NO	$x=3$	9
Minimum	$x=0$	$x=0$	0

(b) $f(x) = x^2$, $-3 \leq x \leq 3$ closed

Cont.

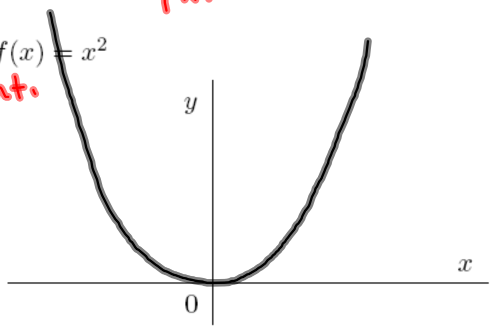


	Local	*Absolute*	*Value*
Maximum	NO	$x=\pm 3$	9
Minimum	$x=0$	$x=0$	0

interval not closed.

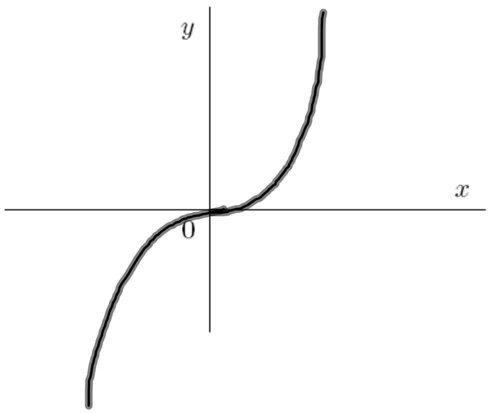
(c) $f(x) = x^2$

Cont.



	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	x=0	x=0	0

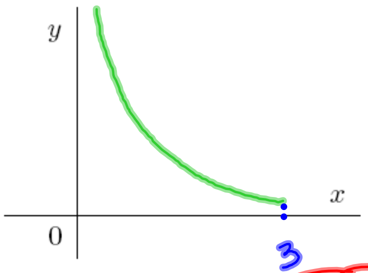
(d) $f(x) = x^3$



	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	NO	NO	NO

cont. not closed

(e) $f(x) = \dfrac{1}{x}$, $0 \leq x \leq 3$

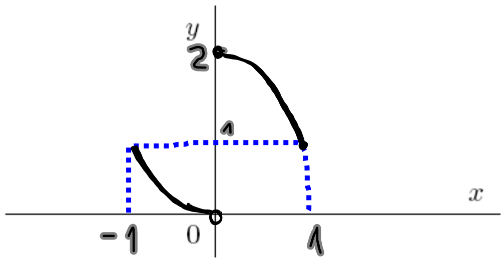


	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	NO	$x=3$	$\frac{1}{3}$

(f) $f(x) = \begin{cases} x^4 & \text{if } -1 \leq x < 0 \\ 2 - x^4 & \text{if } 0 \leq x \leq 1 \end{cases}$ closed interval

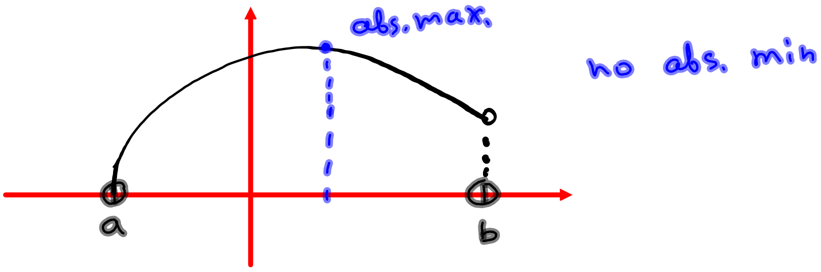
not continuous

	Local	*Absolute*	*Value*
Maximum	NO	$x=0$	2
Minimum	NO	No	NO

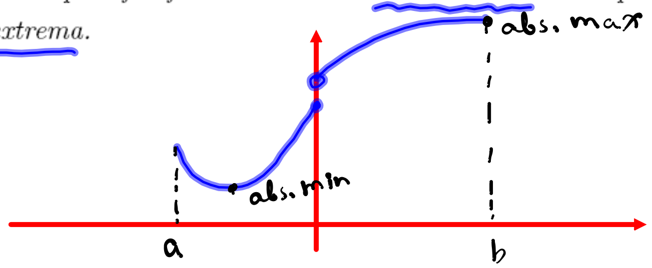


Extreme Value Theorem: If f is a continuous function on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum.

EXAMPLE 4. *Graph an example of a continuous function on a non closed interval that does not attain an an absolute minimum but does attain an absolute maximum.*



EXAMPLE 5. *Graph an example of a function that is not continuous at a point in the given interval and yet has both absolute extrema.*

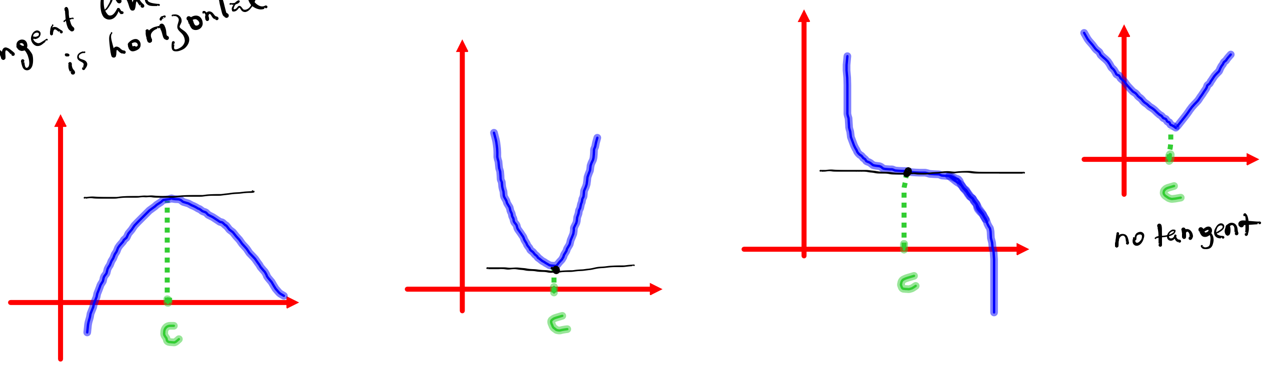


or critical point

DEFINITION 6. *A* **critical number** *of $f(x)$ is a number c is in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.* $\Rightarrow$ no tangent

$\Downarrow$ tangent line is horizontal

Illustration:



x = c is critical point

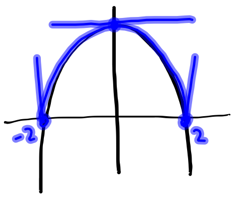
EXAMPLE 7. *Find the critical numbers of $f(x)$:*

(a) $f(x) = x^3 - 3x^2 + 3x$

$$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0$$

$$\boxed{x = 1}$$

(b) $f(x) = |4 - x^2|$



$$\boxed{x = \pm 2} \quad \text{where derivative DNE}$$

$$\boxed{x = 0}$$

Without the graph

$$f(x) = \begin{cases} 4 - x^2, & -2 \le x \le 2 \\ x^2 - 4, & |x| \ge 2 \end{cases}$$

$$\begin{cases} f'_-(2) = -4 \ne f'_+(2) = 4 \\ f'_-(-2) = 4 \ne f'_+(-2) = -4 \end{cases}$$

$\Rightarrow f'(\pm 2)$ DNE

$$f'(x) = \begin{cases} -2x, & |x| \le 2 \\ 2x, & |x| \ge 2 \end{cases}$$

$\Rightarrow f'(x) = 0 \Leftrightarrow x = 0$

EXAMPLE 7. *Find the critical numbers of $f(x)$:*

(c) $f(x) = x^{2/5}(5-x)$ $\Rightarrow f'(x) = \left(5x^{\frac{2}{5}} - x^{\frac{7}{5}}\right)'$

$f'(x) = 5 \cdot \frac{2}{5} x^{\frac{2}{5}-1} - \frac{7}{5} \cdot x^{\frac{7}{5}-1} = 2x^{-\frac{3}{5}} - \frac{7}{5} x^{\frac{2}{5}} = 0$

$x^{-\frac{3}{5}}\left(2 - \frac{7}{5}x\right) = 0$

Critical numbers: $\underset{\text{DNE}}{x=0}$ and $\underset{=0}{x = \frac{10}{7}}$

(d) $f(x) = x \ln x$

$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1} = \boxed{\frac{1}{e}}$ critical number

EXAMPLE 8. *Find the absolute extrema for $f(x)$ on the interval I where*

(a) $\underbrace{f(x) = x^3 - 3x^2 + 3x}_{\text{cont.}}$, $I = \underbrace{[-1, 3]}_{\text{closed}}$

Remark: we don't need classify critical points here

① Find critical points

by Ex. 7(a) $x = 1$ is critical point

② Find value of function at critical points belonging to the given interval.

$x = 1$ belongs to $[-1, 3]$

$f(1) = 1 - 3 + 3 = \boxed{1}$

③ Find values of $f(x)$ at end points:

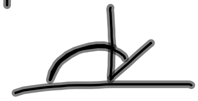
$f(-1) = -1 - 3 - 3 = \boxed{-7}$

$f(3) = 27 - 27 + 9 = \boxed{9}$

④ $\max_{[-1,3]} f(x) = 9$ $\min_{[-1,3]} f(x) = -7$

(b) $f(x) = \underbrace{\sqrt{3}x^2 + 2\cos x^2}_{\text{cont.}}$, $I = \underbrace{\left[\frac{\sqrt{\pi}}{2}, \sqrt{\pi}\right]}_{\text{closed}}$

$$\frac{\sqrt{\pi}}{2} \le x \le \sqrt{\pi}$$
$$\Downarrow$$
$$\frac{\pi}{4} \le x^2 \le \pi$$



$$f'(x) = 2\sqrt{3}\,x - 4x\sin x^2 = 0$$

$$2x\left(\sqrt{3} - 2\sin x^2\right) = 0$$

OR

$\underbrace{x=0}_{\text{crit. point}}$ which is not in I

$$\sqrt{3} = 2\sin x^2$$

$$\sin x^2 = \frac{\sqrt{3}}{2} \Rightarrow x^2 = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\underbrace{x = \pm\sqrt{\frac{\pi}{3}}, \pm\sqrt{\frac{2\pi}{3}}}_{\text{crit. points in } I}$$

Find values of f at crit. points from the given interval and at end points:

$\sqrt{3} \approx 1.7$

$\sqrt{2} \approx 1.4$

$$f\left(\sqrt{\frac{\pi}{3}}\right) = \sqrt{3}\,\frac{\pi}{3} + 2\cos\frac{\pi}{3} = \frac{\sqrt{3}}{3}\pi + 1 \approx \boxed{2.81}$$

$$f\left(\sqrt{\frac{2\pi}{3}}\right) = \sqrt{3}\,\frac{2\pi}{3} + 2\cos\frac{2\pi}{3} = \frac{2\sqrt{3}\pi}{3} - 2\cdot\frac{1}{2} \approx \boxed{2.63}$$

$$f\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{3}\,\frac{\pi}{4} + 2\cos\frac{\pi}{4} \approx \boxed{2.77}$$

$$f\left(\sqrt{\pi}\right) = \sqrt{3}\pi + 2\cos\pi = \sqrt{3}\pi - 2 \approx \boxed{3.44}$$

$$\max_I f = f(\sqrt{\pi}) \approx 3.44$$

$$\min_I f = f\left(\sqrt{\frac{2\pi}{3}}\right) \approx 2.63$$