5.3: Derivatives and Shapes of Curves

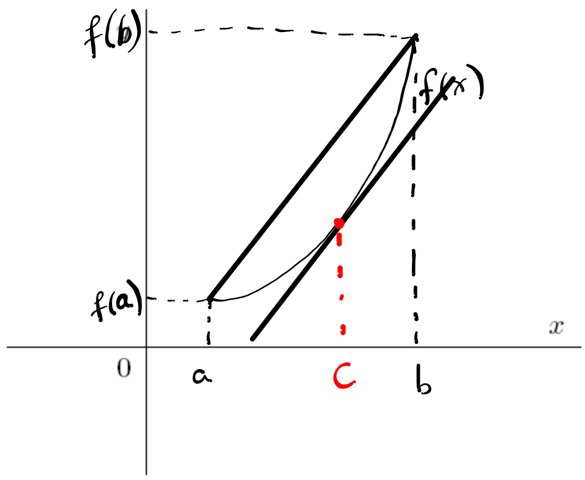
Mean Value Theorem: Suppose a function f is continuous on the (closed) interval $[a, b]$ and differentiable on the (open) interval (a, b). Then there is a number c such that $a < c < b$ and

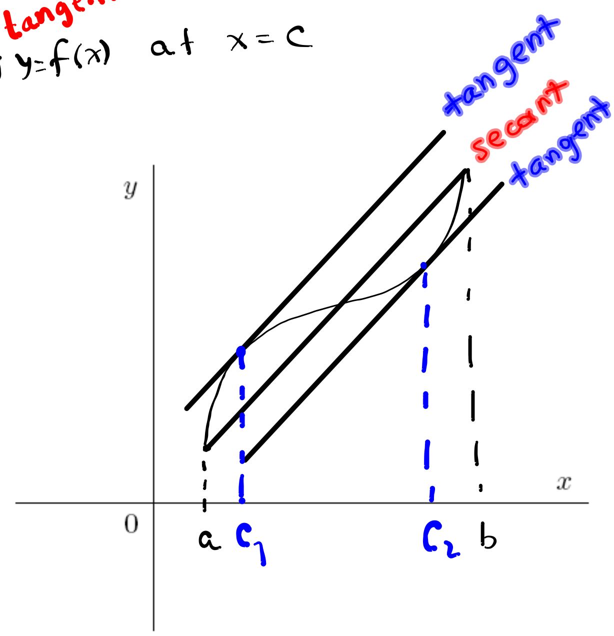
$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Illustration: $m_{AB} = \frac{f(b) - f(a)}{b - a}$ = slope of secant = slope of tangent $y = f(x)$ at $x = c$





EXAMPLE 1. *At* $2:00$ *pm a car's speedometer reads* 30 mi/h. *At* $2:10$ *pm it reads* 50 mi/h. *Show that at some time between* $2:00$ *and* $2:10$ *the acceleration is exactly* $120 \text{mi}/\text{h}^2$.

$a(t) = v'(t)$

by mean value theorem there exists c between a and b s.t

$a = 2:00$

$b = 2:10$pm

$$\frac{v(a) - v(b)}{a - b} = v'(c)$$

$v(a) = 30$ mi/h

$v(b) = 50$ mi/h

$a - b = -10 \text{ min} = -\frac{1}{6}$ h

$$v'(c) = \frac{30 - 50}{-1/6} = 20 \cdot 6 = 120 \text{ mi}/\text{h}^2$$

$$\| $$

$$a(c)$$

EXAMPLE 2. *Find a number c that satisfies the conclusion of the Mean Value Theorem on the interval* $[0, 2]$ *when* $f(x) = x^3 + x - 1$.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(x) = 3x^2 + 1 \Rightarrow f'(c) = 3c^2 + 1$$

$$f(2) = 8 + 2 - 1 = 9$$

$$f(0) = -1$$

$$3c^2 + 1 = \frac{9 - (-1)}{2} = 5$$

$$3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}}$$

belongs to $[0,2)$

Final answer: $\boxed{\frac{2}{\sqrt{3}}}$

EXAMPLE 3. *Suppose* $1 \leq f'(x) \leq 4$ *for all* x *in the [2, 5]. Show that* $3 \leq f(5) - f(2) < 12$.

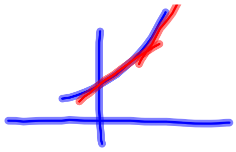
$3 < f(5) - f(2) < 12$

By Mean Value Theorem: $f'(c) = \dfrac{f(5) - f(2)}{5 - 2}$

$3 \cdot 1 \leq f(5) - f(2) = 3f'(c) \leq 3 \cdot 4$

$3 \leq f(5) - f(2) \leq 12$

Test for increasing/decreasing



- If $f'(x) > 0$ on an interval , then f is increasing on that interval.
- If $f'(x) < 0$ on an interval , then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval , then f is constant on that interval.

EXAMPLE 4. *Determine all intervals where the following function*

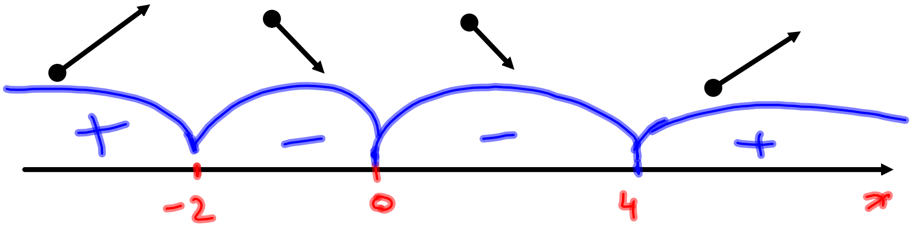
$$f(x) = x^5 - \frac{5}{2}x^4 - \frac{40}{3}x^3 - 12$$

is increasing or decreasing.

even power

$f'(x) = 5x^4 - 10x^3 - 40x^2$

$f'(x) = 5x^2(x^2 - 2x - 8) = 5x^2(x-4)(x+2)$

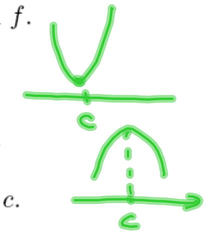


$(-\infty, -2) \cup (4, \infty)$ increasing

$(-2, 4)$ decreasing

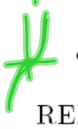
$f'(c)=0$ OR $f'(c)$ DNE

First Derivative Test: Suppose that $x = c$ is a critical point of a continuous function f.



- If $f'(x)$ changes from negative to positive at $x = c$, then f has a local minimum at c.
- If $f'(x)$ changes from positive to negative at $x = c$, then f has a local maximum at c.

$y=x^3$



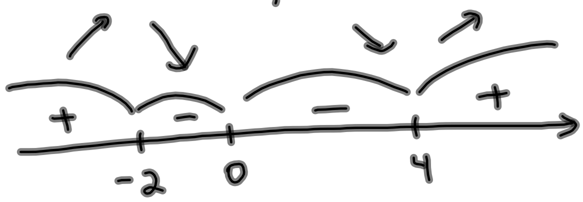
no local extremum

- If $f'(x)$ does not change sign at $x = c$, then f has no local maximum or minimum at c.

REMARK 5. The first derivative test only classifies critical points as local extrema and not as absolute extrema.

EXAMPLE 6. *For function from Example 4 identify all local extrema.*

In $\underline{\text{Ex. 4}}$ $f'(x) = 0 \iff x = 0,\ x = 4,\ x = -2$



$x = -2$ local max

$x = 0$ no local extremum

$x = 4$ local min

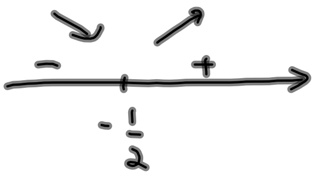
EXAMPLE 7. *Find all intervals of increase and decrease of f and identify all local extrema.*

(a) $f(x) = xe^{2x}$

$$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}(1+2x) = 0 \Rightarrow 1+2x=0$$

$\Rightarrow x = -\frac{1}{2}$ is critical number

local minimum



(b) $f(x) = x\sqrt[3]{x^2 - \frac{5}{3}} = \sqrt[3]{x^3\left(x^2 - \frac{5}{3}\right)} = \Big(\overbrace{x^5 - \frac{5}{3}x^3}^{\text{inner}}\Big)^{1/3}$

$$f'(x) = \frac{1}{3}\left(x^3\left(x^2 - \frac{5}{3}\right)\right)^{-\frac{2}{3}}\left(5x^4 - 5x^2\right)$$

$$f'(x) = \frac{5\cancel{x^2}(x^2-1)}{3\cancel{x^2}\sqrt[3]{\left(x^2 - 5/3\right)^2}} = 0 \Rightarrow$$

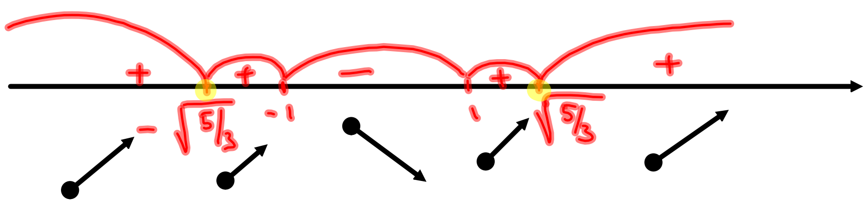
~~$x = 0$~~

$x = 1$

$x = -1$

$f'(x)$ DNE $\begin{cases} x = \sqrt{\frac{5}{3}} \\ x = -\sqrt{\frac{5}{3}} \end{cases}$

$$f'(x) = \frac{5(x-1)(x+1)}{3\sqrt[3]{\left(x-\sqrt{\frac{5}{3}}\right)^2\left(x+\sqrt{\frac{5}{3}}\right)^2}}$$



$x = -1$ local max

$x = 1$ local min

$x = \pm\sqrt{\frac{5}{3}}$ inflection points (see definition below)

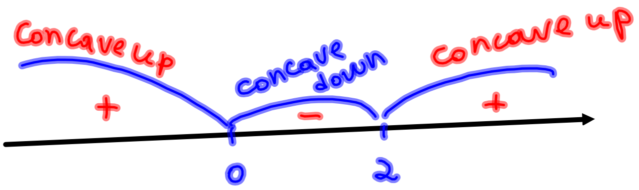
Recall here the **Second derivative test for concavity.** (see Section 5.1):

- If $f''(x) > 0$ for all x on an interval, then f is concave up on that interval.
- If $f''(x) < 0$ for all x on an interval, then f is concave down on that interval.

In addition, if f changes concavity at $x = a$, and $x = a$ is in the domain of f, then $x = a$ is an inflection point of f.

EXAMPLE 8. *Find intervals of concavity and inflection points of f, if $f'(x) = 4x^3 - 12x^2$.*

$f''(x) = 12x^2 - 24x = 12x(x-2)$



$x=0$ and $x=2$ are inflection points because $f(x)$ changes concavity there.

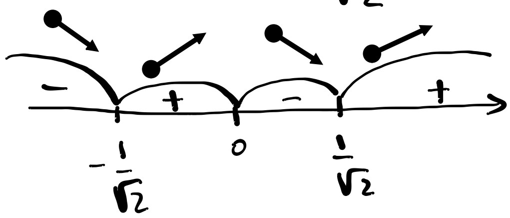
EXAMPLE 9. *Sketch the graph of f by locating intervals of increase/decrease, local extrema, concavity and inflection points.*

(a) $f(x) = x^4 - x^2$

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 2x(\sqrt{2}x - 1)(\sqrt{2}x + 1) = 0$$

Crit. numbers: $x = 0$, $x = \frac{1}{\sqrt{2}}$, $x = -\frac{1}{\sqrt{2}}$

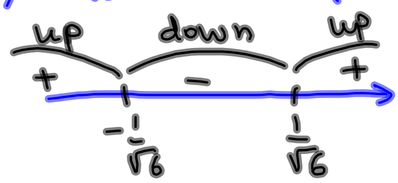
sign of f'



increas./decreas.

Concavity

$$f''(x) = 12x^2 - 2 = 2(6x^2 - 1) = 2(\sqrt{6}x - 1)(\sqrt{6}x + 1) = 0$$

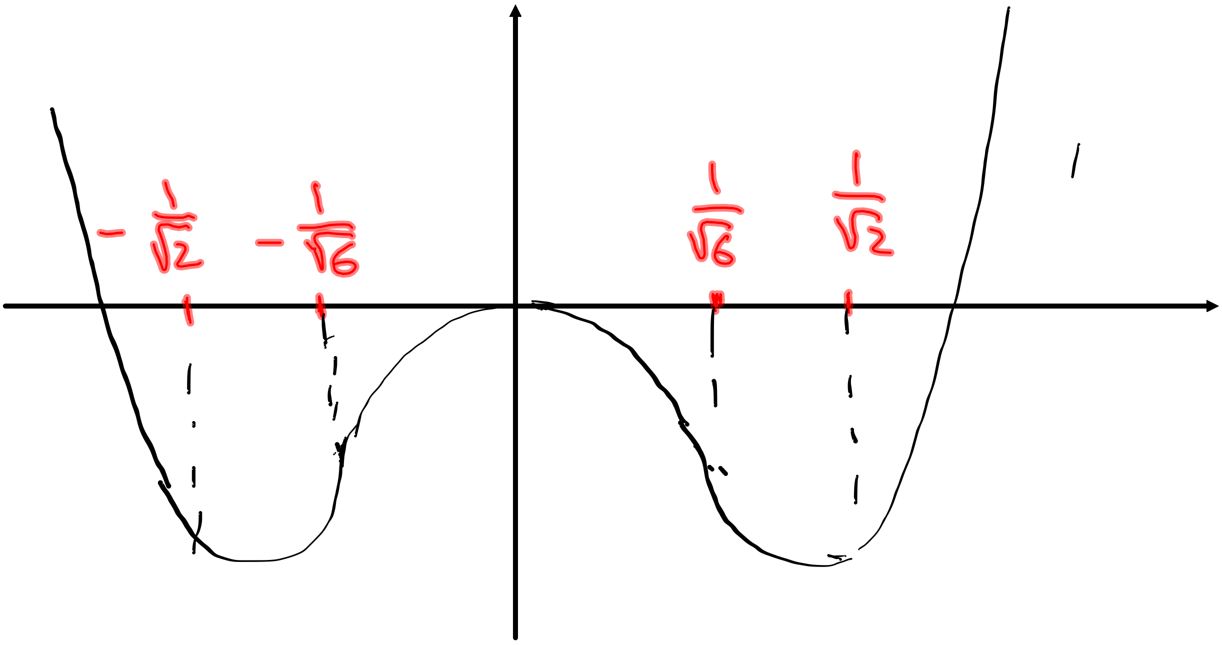


$x = \pm\frac{1}{\sqrt{6}}$

inflection points

Ex. 9(a) continued

Also note $f(0) = 0$



(b) $f(x) = \dfrac{x}{(x-1)^2}$

Due Wednesday
at the beginning of class

$x = c$ is critical number

Second derivative test for local extrema: Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

REMARK 10. If $f'(c) = 0$ and $f''(c) = 0$ or does not exist, then the test fails. In the case $f''(c)$ does not exist we use the first derivative test to find the local extrema.

EXAMPLE 11. *Find the local extrema for* $f(x) = 1 - 3x + 5x^2 - x^3$.

$$f'(x) = -3 + 10x - 3x^2 = -3\left(x^2 - \frac{10}{3}x + 1\right)$$

$$f'(x) = -3(x-3)\left(x - \frac{1}{3}\right) = 0$$

critical numbers: $x = 3$ and $x = \frac{1}{3}$

$$f''(x) = 10 - 6x = 2(5 - 3x)$$

$f''(3) = 2(5 - 3\cdot 3) < 0 \Rightarrow x = 3$ is local max

$f''\left(\frac{1}{3}\right) = 2\left(5 - 3\cdot\frac{1}{3}\right) > 0 \Rightarrow x = \frac{1}{3}$ is local min