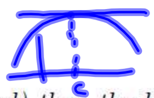
5.5: Applied Maximum and Minimum Problems (determine abs. extrema on non-closed interval)

OPTIMIZATION PROBLEMS

First derivative test for absolute extrema: Suppose that c is a critical number of a continuous function f defined on an interval. ⇒ $f'(c)=0$

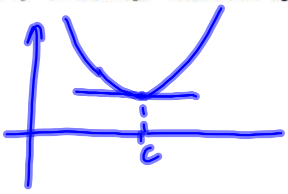
unique crit. number

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.



Alternatively,

- *If $f''(x) < 0$ for all x (so f is always concave downward) then the local maximum at c must be an absolute maximum.*
- *If $f''(x) > 0$ for all x (so f is always concave upward) then the local minimum at c must be an absolute minimum.*



$x \geq 0$

EXAMPLE 1. *When a producer sells x items per week, he makes a profit of*

$$p(x) = 15x - 0.001x^2 - 2000.$$

How many items does he need to sell to get the maximum profit?

Find abs. max of $p(x)$ for all $x \geq 0$.

Look for critical numbers:

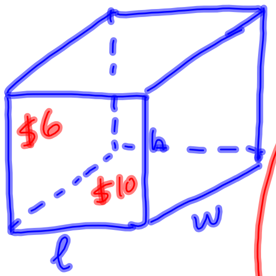
$p'(x) = 0 \iff 15 - 0.002x = 0$

$x = \dfrac{15}{0.002} = \underbrace{7500}_{\text{unique crit. number}}$ items

$p''(x) = -0.002 < 0$ for all $x \Rightarrow$

$\Rightarrow$ at $x = 7500$ p has is absolute maximum.

EXAMPLE 2. *A rectangular storage container with an ~~open top~~ is to have a volume of* 10m^3. *The length of its base is twice the width. Material for the base costs \$10 per squire meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.*



$$\begin{cases} V = 10 = \ell w h \\ \ell = 2w \end{cases}$$

$$C = 10 \cdot \ell w + 6 \cdot 2(\ell h + w h)$$

$$C = 10 \ell w + 12 h (\ell + w)$$

Eliminate to parameters using those 2 conditions

$$\ell = 2w, \quad h = \frac{10}{\ell w} = \frac{10}{2w \cdot w} = \frac{5}{w^2}$$

$$C(w) = 10 \cdot 2w \cdot w + 12 \cdot \frac{5}{w^{\cancel{2}}} \cdot (2\cancel{w} + \cancel{w})$$

$$C(w) = 20w^2 + \frac{\overset{3 \cdot 4 \cdot 5 \cdot 3}{12 \cdot 5 \cdot 3}}{w} = 20(w^2 + 9w^{-1})$$

where $0 < w$ (open interval)

Critical numbers:

$$C'(w) = 20(2w - 9w^{-2}) = 0$$

$$2w = 9w^{-2}$$

$$2w^3 = 9 \Rightarrow w^3 = \frac{9}{2} \Rightarrow w = \underbrace{\sqrt[3]{\frac{9}{2}}}_{\text{belongs to } (0, \infty)}$$

unique critical number on the considered interval

$$C''(w) = 20(2 + 9 \cdot 2 \cdot w^{-3}) = 40\left(1 + \frac{9}{w^3}\right) > 0$$

for ALL $w > 0$

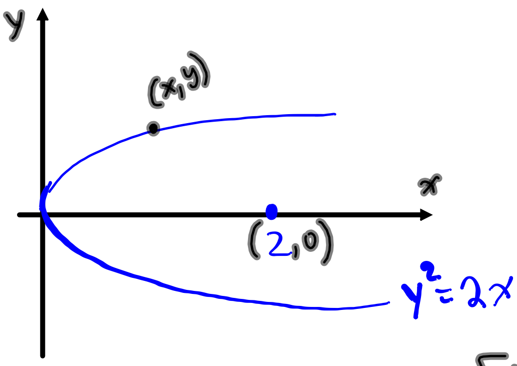
$\Rightarrow \quad w = \sqrt[3]{\frac{9}{2}}$ is point of abs. minimum

Cost of material for cheapest such container is

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\left(\sqrt[3]{\frac{9}{2}}\right)^2 + 9 \cdot \sqrt[3]{\frac{2}{9}}\right) \approx$$

$$\approx \$163.54$$

EXAMPLE 3. *Find the shortest distance from the parabola* $y^2 = 2x$ *to the point* $(2, 0)$.



$$d = \sqrt{(2-x)^2 + (0-y)^2}$$

$$= \sqrt{4-4x+x^2+y^2} = \sqrt{4-4x+x^2+2x}$$

$$d(x) = \sqrt{4-2x+x^2}$$

Find $\min\limits_{x \geq 0} d(x) = \sqrt{\min\limits_{x \geq 0} d^2(x)}$

$$f(x) = d^2(x) = 4 - 2x + x^2$$

$f'(x) = -2 + 2x = 0 \Rightarrow x = 1$ unique critical number on $[0, \infty)$

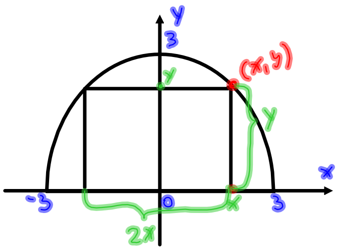
$f''(x) = 2 > 0$ for all x

Conclusion $f(x)$ has abs. minimum at $x=1$ on $[0,\infty)$

$$f(1) = 4 - 2 + 1 = 3$$

$$\min_{0 \leq x < \infty} d(x) = \sqrt{3}$$

EXAMPLE 4. *A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$. What length and width should the rectangle have so that its area is a maximum?* (Equivalently, find the dimensions of the largest rectangle that can be inscribed in the semi-disk with radius 3.)



$$\begin{cases} y = \sqrt{9-x^2} \\ A = 2xy \end{cases} \Rightarrow A(x) = 2x\sqrt{9-x^2}$$

$$\max_{0<x<3} A(x) = ?$$

Find critical numbers

$$A'(x) = 2\left(x\sqrt{9-x^2}\right)' = 2\left(\sqrt{9-x^2} + x\frac{-2x}{2\sqrt{9-x^2}}\right)$$

$$A'(x) = 2\left(\sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}}\right) = \frac{2}{\sqrt{9-x^2}}\left(9-x^2-x^2\right)$$

$$A'(x) = \frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$$

$$\Rightarrow 9 = 2x^2 \Rightarrow x = \pm\sqrt{\frac{9}{2}} = + \frac{3}{\sqrt{2}}$$

Note that $A(x)$ has 4 critical numbers $x = \pm\frac{3}{\sqrt{2}}, \pm 3$

However, only $x = \frac{3}{\sqrt{2}}$ belong to $(0,3)$.

$$A''(x) = \frac{2\left[-4x\sqrt{9-x^2} - (9-2x^2)\cdot\frac{-2x}{2\sqrt{9-x^2}}\right]}{9-x^2}$$

$$A''(x) = \frac{\cancel{2}\left[-4x(9-x^2) + 2x(9-2x^2)\right]}{\cancel{2}(9-x^2)\sqrt{9-x^2}} =$$

$$A''(x) = \frac{-72x + 8x^3 + 18x - 4x^3}{(9-x^2)^{3/2}} = \frac{4x^3 - 54x}{(9-x^2)^{3/2}}$$

$$A''(x) = \frac{2x(2x^2-27)}{(9-x^2)^{3/2}} < 0 \text{ for all } 0 < x < 3.$$

$\Rightarrow$ at $x = \frac{3}{\sqrt{2}}$ the abs. max is attained.

The dimensions of the largest rectangle inscribed in the semicircle $y = \sqrt{9-x^2}$ are

$$2x = \frac{3\cdot 2}{\sqrt{2}} = \frac{6}{\sqrt{2}} \text{ and } y = \sqrt{9 - \left(\frac{3}{\sqrt{2}}\right)^2} = \sqrt{9 - \frac{9}{2}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$

Final answer: $\boxed{\frac{6}{\sqrt{2}} \times \frac{3}{\sqrt{2}}}$