Section 5.7: Antiderivatives

DEFINITION 1. *A function F is called an* **antiderivative** *of f on an interval I if $F'(x) = f(x)$ for all x in I.*

EXAMPLE 2. **(a)** *Is the function $F(x) = x\ln(x) - x + \sin x$ is an antiderivative of $f(x) = \ln(x) + \cos x$?*

$F'(x) = \ln x + x \cdot \frac{1}{x} - 1 + \cos x = \ln x + \cos x$

YES!

(b) *Is the function $F(x) = x\ln(x) - x + \sin x + 10$ is an antiderivative of $f(x) = \ln(x) + \cos x$?*

$(10)' = 0 \Rightarrow$ YES!

(c) *What is the most general antiderivative of $f(x) = \ln(x) + \cos x$?*

$F(x) = \ln x + \cos x + \text{const}$

$F'(x) = f(x)$ for all x.

THEOREM 3. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $\underbrace{F(x) + C}$, where C is an arbitrary constant.*

EXAMPLE 4. *Find the most general antiderivative of $f = 2x$.*

$F(x) = x^2 + C$

Table of Antidifferentiation Formulas

$f(x)$ $\qquad$ $F(x)$ $\qquad$ $F(x) + C$ $\qquad$ $f(x) = (F(x)+C)' = F'(x)$

Function	Particular antiderivative	Most general antiderivative
$k \quad (k \in \mathbb{R})$	kx	$kx + C$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	
e^x	e^x	
$\cos x$	$\sin x$	
$\sin x$	$-\cos x$	$+ C$
$\sec^2 x$	$\tan x$	
$\csc^2 x$	$-\cot x$	
$\sec x \tan x$	$\sec x$	
$\csc x \cot x$	$\csc x$	
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	
$\dfrac{1}{1+x^2}$	$\arctan x$	

EXAMPLE 5. *Find the most general antiderivative of f where*

(a) $f(x) = 5\sin x + 7x^6 - \sqrt[8]{x^7} + 15$ $= 5\sin x + 7x^6 - x^{7/8} + 15$

$$F(x) = 5(-\cos x) + 7\frac{x^{6+1}}{6+1} - \frac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + 15x + C$$

$$F(x) = -5\cos x + x^7 - \frac{8}{15}x^{15/8} + 15x + C$$

$$\left.\begin{array}{l} F'(x) = f(x) \\ G'(x) = g(x) \\ \left(\frac{F}{G}\right)' = h(x) \end{array}\right\} \not\Rightarrow h(x) = \frac{f(x)}{g(x)}$$

(b) $f(x) = \dfrac{3x + 8 - x^2}{x^3}$

$$f(x) = 3x^{-2} + 8x^{-3} - \frac{1}{x}$$

$$F(x) = 3\frac{x^{-2+1}}{-2+1} + 8\frac{x^{-3+1}}{-3+1} - \ln|x| + C$$

$$F(x) = -\frac{3}{x} - \frac{4}{x^2} - \ln|x| + C$$

(c) $f(x) = e^x + (1 - x^2)^{-1/2}$

$$f(x) = e^x + \frac{1}{\sqrt{1-x^2}} \Rightarrow F(x) = e^x + \arcsin x + C$$

EXAMPLE 6. *Find $f(x)$ given that $f'(x) = 4 - 3(1+x^2)^{-1}$, $\quad f(1) = 0$.*

$$f'(x) = 4 - \frac{3}{1+x^2}$$

$$f(x) = 4x - 3\arctan x + C$$

$$f(1) = 0 \iff 4 - 3\arctan 1 + C = 0$$

$$C = -4 + 3\arctan 1 = -4 + \frac{3\pi}{4}$$

$$f(x) = 4x - 3\arctan x - 4 + \frac{3\pi}{4}$$

EXAMPLE 7. *Find* $f(x)$ *given that* $\underline{f''(x) = 3e^x + 5sinx}$, $f(0) = 1$, $f'(0) = 2$.

$$f'(x) = 3e^x - 5\cos x + C$$

$$f(x) = 3e^x - 5\sin x + Cx + C_1$$

$$1 = f(0) = 3 + C_1 \quad \Rightarrow \quad C_1 = -2$$

$$2 = f'(0) = 3 - 5 + C = C - 2 \quad \Rightarrow \quad C = 4$$

$$\boxed{f(x) = 3e^x - 5\sin x + 4x - 2}$$

EXAMPLE 8. *A particle is moving according to acceleration $a(t) = 2t + 2$. Find the position, $s(t)$, of the object at time t if we know $s(0) = 1$ and $v(0) = -2$.*

Find $s(t)$ if

$$a(t) = s''(t) = 2t+2, \quad \underbrace{s(0)=1,\ v(0)=s'(0)=-2}_{\text{use them to find } c_1 \,\&\, c}$$

$$\Rightarrow \quad s'(t) = t^2 + 2t + C$$

$$s(t) = \frac{t^3}{3} + t^2 + Ct + C_1$$

$1 = s(0) = C_1 \quad ; \quad -2 = s'(0) = C$

$$\boxed{s(t) = \frac{t^3}{3} + t^2 - 2t + 1}$$

EXAMPLE 9. *Show that for motion in a straight line with a constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is*

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0.$$

$s''(t) = a$, $v_0 = s'(0)$, $s_0 = s(0)$

$\Downarrow$

$s'(t) = at + C$

$\Downarrow$

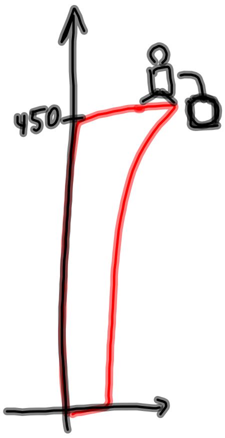
$s(t) = \frac{1}{2}at^2 + Ct + C_1$

$v_0 = s'(0) = C \Rightarrow s'(t) = at + v_0$

$s_0 = s(0) = C_1$

$$\boxed{s(t) = \frac{1}{2}at^2 + v_0 t + s_0}$$

EXAMPLE 10. *A stone is thrown downward from a 450 m tall building at a speed of 5 meters per second. Derive a formula for the distance of the stone above ground level.*



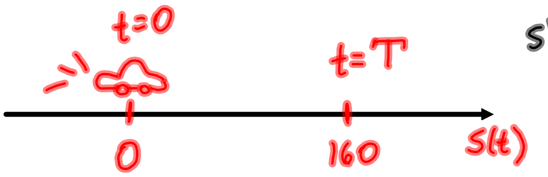
$s''(t) = a = -9.8 \qquad s(0) = 450 \qquad v_0 = -5 = s'(0)$

By Ex. 9

$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$

$s(t) = -\frac{9.8}{2}t^2 - 5t + 450$

EXAMPLE 11. *A car braked with a constant deceleration of* 40ft/s^2, *producing skid marks measuring* 160ft *before coming to a stop. How fast was the car traveling when the brakes were first applied?*



$s''(t) = a = -40 \text{ ft/s}^2$

$s(T) = 160 \text{ ft} \quad s_0 = s(0) = 0$

$v_0 = v(0) = ?$

$v(T) = 0$

$s''(t) = -40$

$s'(t) = -40t + v_0 \quad \Rightarrow \quad 0 = v(T) = s'(T) = -40T + v_0$

$s(t) = -20t^2 + v_0 t + s_0 \quad 160 = s(T) = -20T^2 + v_0 T + 0$

$$\begin{cases} -40T + v_0 = 0 \\ -20T^2 + v_0 T = 160 \end{cases} \Rightarrow T = \frac{v_0}{40}$$

$$-\cancel{20}\cdot\frac{v_0^2}{\cancel{40^2}_{\,40\cdot 2}} + \frac{{}^{2\cdot}\!/\,v_0\cdot v_0}{40} = 160$$

$$\frac{-v_0^2 + 2v_0^2}{40\cdot 2} = 160$$

$$v_0^2 = 160\cdot 40\cdot 2$$

$$v_0^2 = 40\cdot 40\cdot 2^2\cdot 2$$

$$v_0 = +40\cdot 2\sqrt{2} = 80\sqrt{2}$$

$$v_0 \approx 113 \text{ ft/s}$$