6.1: Sigma notation

Σ sigma

DEFINITION 1. *If* $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ *are real numbers and* m *and* n *are integers such that* $m \leq n$, *then*

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n.$$

$$\sum_{k=1}^{3}(a_k - a_{k+1}) = a_1 - \cancel{a_2} + \cancel{a_2} - \cancel{a_3} + \cancel{a_3} - a_4 = a_1 - a_4$$

Telescoping sum

EXAMPLE 2. *Compute these summations:*

(a) $\sum_{i=5}^{8}(\sin i - \sin(i+1)) =$

$$\underbrace{\sin 5 - \cancel{\sin 6}}_{i=5} + \cancel{\sin 6} - \cancel{\sin 7} + \cancel{\sin 7} - \cancel{\sin 8} + \underbrace{\cancel{\sin 8} - \sin 9}_{i=8} = \boxed{\sin 5 - \sin 9}$$

(b) $\sum_{i=1}^{10} 3 = \underset{i=1}{3} + \underset{i=2}{3} + \ldots + \underset{i=10}{3} = 3 \cdot 10 = 30$

EXAMPLE 3. *Write the sum in sigma notation:*

$-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} = -\underset{k=1}{\frac{1}{1^2}} + \underset{k=2}{\frac{1}{2^2}} - \frac{1}{3^2} + \frac{1}{4^2} - \underset{k=5}{\frac{1}{5^2}} + \ldots$

$$= \sum_{k=1}^{5} \frac{(-1)^k}{k^2}$$

THEOREM 4. *If c is any constant then*

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Note that in general

$$\sum_{i=m}^{n} a_i b_i \neq \left(\sum_{i=m}^{n} a_i\right) \cdot \left(\sum_{i=m}^{n} b_i\right).$$

$a_1b_1 + a_2b_2 \neq (a_1 + a_2)(b_1 + b_2)$

EXAMPLE 5. *If* $\sum_{i=1}^{25} f(i) = 15$, $f(25) = 7$ *and* $\sum_{i=1}^{24} g(i) = 25$ *find*

$$\sum_{i=1}^{24} (2f(i) - g(i)) = 2\sum_{i=1}^{24} f(i) - \sum_{i=1}^{24} g(i) = 2\sum_{i=1}^{24} f(i) - 25 =$$

$$= 2\cdot 8 - 25 = \boxed{-9}$$

$$15 = \sum_{i=1}^{25} f(i) = \sum_{i=1}^{24} f(i) + \underbrace{f(25)}_{7} \Rightarrow \sum_{i=1}^{24} = 15 - 7 = 8$$

THEOREM 6.

- $\displaystyle\sum_{i=1}^{n} 1 = n$
- $\displaystyle\sum_{i=1}^{n} c = nc,$ *where c is a constant.*
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 7. *Compute these sums:*

(a) $\sum_{i=4}^{27} 20 =$ $\sum_{i=1}^{27} 20 - \sum_{i=1}^{3} 20 = 20\cdot 27 - 20\cdot 3 = 20(27-3) =$

$= 20\cdot 24 = \boxed{480}$

(b) $\sum_{i=1}^{n} i(i+4) =$ $\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 4i = \frac{n(n+1)(2n+1)}{6} + 4\cdot\sum_{i=1}^{n} i$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{\overset{2}{\cancel{4}}(n+1)n}{\cancel{2}}$$

$k=1$

(c) $\displaystyle\sum_{k=1}^{100}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} +$

TELESCOPING

$+ \ldots + \cancel{\frac{1}{99}} - \frac{1}{101} = 1 - \frac{1}{101} = \frac{100}{101}$

EXAMPLE 8. *Find the limit:* $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n}\left[\left(\frac{j}{n}\right)^3 + 1\right]$

$$\sum_{j=1}^{n} \frac{1}{n}\left[\left(\frac{j}{n}\right)^3 + 1\right] = \frac{1}{n}\left[\sum_{j=1}^{n} \frac{j^3}{n^3} + \sum_{j=1}^{n} 1\right] =$$

$$= \frac{1}{n}\left[\frac{1}{n^3}\sum_{j=1}^{n} j^3 + n\right] = \frac{1}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 + 1 \xrightarrow[n\to\infty]{} \frac{1}{4}+1 = \boxed{\frac{5}{4}}$$