6.3: The Definite Integral

DEFINITION 1. *The* **definite integral of** f **from** a **to** b *is*

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. Here P is a partition of the interval $[a,b]$, $\Delta x_i = (b-a)/n$, and x_i^* is any point in the ith subinterval. If the limit does exist, then f is called **integrable** *on the interval $[a,b]$.*

EXAMPLE 2. *Express the limit as a definite integral:*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\underbrace{\left(1+\frac{2i}{n}\right)}_{x_i^*}^5 - 6\right]\underbrace{\frac{2}{n}}_{\Delta x} = \int_1^3 (3x^5 - 6)\,dx$$

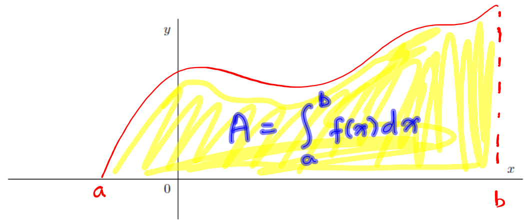
$$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b-a = 2$$

$$x_i^* = x_i = 1 + \frac{2i}{n} = 1 + i\Delta x \Rightarrow a = 1$$

$$\parallel\ a + i\Delta x \qquad b = a+2 = 3$$

The limit represent area under the curve $y = f(x) = 3x^5 - 6$ on the interval $[1,3]$.

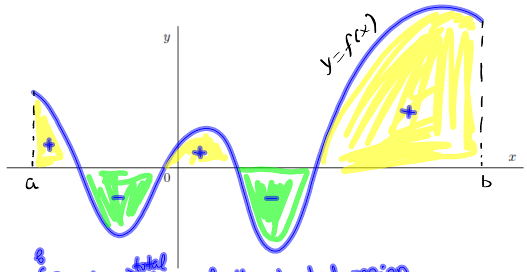
If $f(x) > 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function f and the lines $y = 0$, $x = a$ and $x = b$.



In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

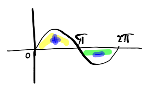
where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the region below the x and above the graph of f.



$\int_a^b f(x)\,dx \neq$ total area of the shaded region

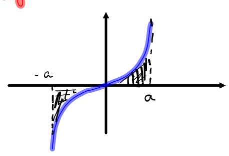
$=$ total area of yellow region − total area of green region

For example $\int_0^{2\pi} \sin x\,dx = 0$



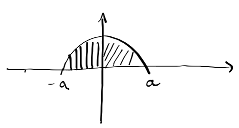
Another example $f(x)$ is an odd function (i.e. $f(-x) = -f(x)$) on a symmetric interval $[-a, a]$

We know that the graph of such function is symmetric w.r.t. origin $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$



If $f(x)$ is an even function (i.e. $f(-x) = f(x)$) on a symmetric interval $[-a, a]$ then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$



because graph of even function is symmetric w.r.t. the y-axis.

Conclusion $\int_{-a}^{a} x\,dx = \int_{-a}^{a} x^3\,dx = \int_{-a}^{a} x^{2n+1}\,dx = 0$

$$\int_{-a}^{a} x^{2n}\,dx = 2\int_0^a x^{2n}\,dx$$

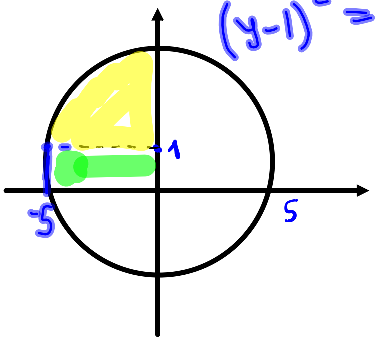
EXAMPLE 3. *Evaluate the following integrals by interpreting each in terms of areas:*

(a) $\int_{-5}^{0} (1+\sqrt{25-x^2})\,dx$

$y = f(x) = 1 + \sqrt{25-x^2} > 0$ on $[-5, 0]$

$y - 1 = \sqrt{25-x^2}$

$(y-1)^2 = 25 - x^2 \Rightarrow x^2 + (y-1)^2 = 25$ circle

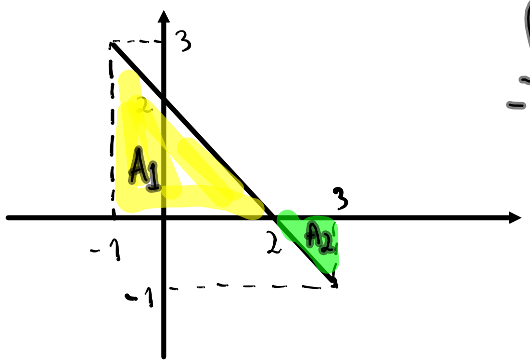


$\int_{-5}^{0} (1+\sqrt{25-x^2})\,dx = \text{Area}\left(\tfrac{1}{4} \text{ or circle } r=5\right) + \text{Area}\left(\text{rectangle } 5\times 1\right)$

$= \frac{1}{4}\pi 5^2 + 5\times 1 = \boxed{\frac{25\pi}{4} + 5}$

(b) $\int_{-1}^{3} \underbrace{(2-x)} \, dx$

$y = f(x) = 2 - x$ on $[-1, 3]$



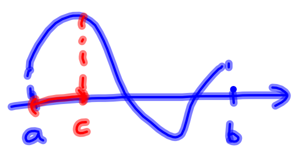
$$\int_{-1}^{3} (2-x)\,dx = A_1 - A_2$$

$$= \frac{1}{2} \cdot 3 \cdot 3 - \frac{1}{2} \cdot 1 \cdot 1 =$$

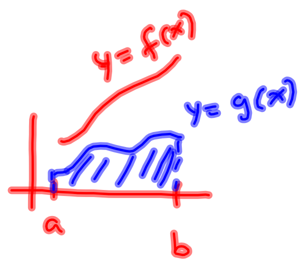
$$= \frac{9}{2} - \frac{1}{2} = \boxed{4}$$

Properties of Definite Integrals:

- $\int_a^b \mathrm{d}x = b - a = \int_a^b 1 \cdot dx =$ length of $[a, b]$.
- $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where c is any constant
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \le c \le b$



- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge 0$
- If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$



- If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$.

EXAMPLE 4. *Write as a single integral:*

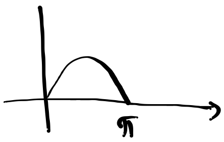
$$\int_{+3}^{5} f(x)\,\mathrm{d}x + \int_{0}^{+3} f(x)\,\mathrm{d}x - \int_{6}^{5} f(x)\,\mathrm{d}x + \int_{5}^{5} f(x)\,\mathrm{d}x$$

$$= \int_{3}^{5} f\,dx + \int_{0}^{3} f\,dx + \int_{5}^{6} f\,dx + 0$$

$$= \int_{0}^{3} + \int_{3}^{5} + \int_{5}^{6} = \int_{0}^{6} f(x)\,dx$$

EXAMPLE 5. *Estimate the value of* $\int_0^{\pi} (4\sin^5 x + 3)\,dx$

$0 \le \sin x \le 1$



$0 \le \sin^5 x \le 1$

$0 \le 4\sin^5 x \le 4$

$\underbrace{3}_{m} \le 4\sin^5 x + 3 \le \underbrace{7}_{M}$

$b - a = \pi - 0 = \pi$

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \Rightarrow$

$$\boxed{3\pi \le \int_0^{\pi} (4\sin^5 x + 3)\,dx \le 7\pi}$$