6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

EXAMPLE 1. *Differentiate* $g(x) = \int_{-4}^{x} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t)}\,dt$

$$g'(x) = f(x) = e^{2x}\cos^2(1-5x)$$

EXAMPLE 2. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\underbrace{\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right)}_{g(u(x))} = f(u(x))u'(x).$$

$$\frac{d}{dx}\left(g(u(x))\right) = g'(u)\cdot u'(x) =$$

$$= \frac{d}{dx}\left(\int_a^u f(t)\,dt\right) u'(x) = f(u)\, u'(x)$$

Let $u(x)$ and $v(x)$ be differentiable functions and $f(x)$ be a continuous one. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x) - f(v(x))v'(x).$$

$$= \frac{d}{dx}\left(\int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right) =$$

$$\frac{d}{dx}\left(-\int_{a}^{v(x)} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right) =$$

$$= -f(v(x))v'(x) + f(u(x))u'(x)$$

EXAMPLE 3. *Differentiate* $g(x)$ *if*

(a) $g(x) = \int_{-4}^{x^3} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t)}\,\mathrm{d}t$

$u(x) = x^3 \Rightarrow u'(x) = 3x^2$

$$\frac{dg}{dx} = f(u)\,u'(x)$$

$$= e^{2x^3}\cos^2(1-5x^3)\cdot 3x^2$$

(b) $g(x) = \int_{e^{x^2}}^{1} \frac{t+1}{\ln t + 3}\,dt$ $= \int_{1}^{e^{x^2}} \underbrace{-\frac{t+1}{\ln t+3}}_{f(t)}\,dt$

$$u(x) = e^{x^2} \Rightarrow u'(x) = 2x\,e^{x^2}$$

$$\frac{dg}{dx} = f(u)\,u'(x) = -\frac{e^{x^2}+1}{\ln e^{x^2}+3}\,2x e^{x^2}$$

$$= -\frac{(e^{x^2}+1)\,2xe^{x^2}}{x^2+3}$$

(c) $g(x) = \int_{x^2}^{\sin x} \underbrace{\frac{\cos t}{t}}_{f(t)}\, \mathrm{d}t$

$u(x) = \sin x \Rightarrow u'(x) = \cos x$

$v(x) = x^2 \Rightarrow v'(x) = 2x$

$$\frac{dg}{dx} = f(u)u'(x) - f(v)\, v'(x)$$

$$= \frac{\cos(\sin x)}{\sin x} \cos x - \frac{\cos x^2}{x^2} \cdot 2x$$

PART II If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

Newton-Leibnitz formula

$$F(x)+C\Big|_a^b = F(b)+\cancel{C}-(F(a)+\cancel{C})$$

$$= F(b)-F(a)$$

$$F'(x) = f(x)$$

$$\int f(x)\,dx = \int F'(x)\,dx = F(x)+C$$

antiderivative

EXAMPLE 4. *Evaluate*

1. $\displaystyle\int_1^5 \frac{1}{x^2}\,\mathrm{d}x = \int_1^5 x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Bigg|_1^5 = -\frac{1}{x}\Bigg|_1^5$

$$= -\left(\frac{1}{5} - \frac{1}{1}\right) = -\left(-\frac{4}{5}\right) = \boxed{\frac{4}{5}}$$

2. $\displaystyle\int_{-\pi/2}^{0} (\cos x - 4\sin x)\,\mathrm{d}x = \sin x + 4\cos x \Big|_{-\frac{\pi}{2}}^{0}$

$$= \sin 0 + 4\cos 0 - \left(\sin\left(-\frac{\pi}{2}\right) + 4\cos\frac{\pi}{2}\right)$$

$$= 0 + 4 - (-1 + 0) = 5$$

3. $\int_0^1 (u^3+2)^2\,\mathrm{d}u = \int_0^1 \left(u^6 + 4u^3 + 4\right) du$

$$= \frac{u^{6+1}}{6+1} + 4\,\frac{u^{3+1}}{3+1} + 4u \Big|_0^1$$

$$= \frac{u^7}{7} + u^4 + 4u \Big|_0^1 = \frac{1}{7} + 1 + 4 = \frac{36}{7}$$

EXAMPLE 5. *Evaluate*

1. $\displaystyle\int_1^2 \frac{2x^5 - x + 3}{x^2}\,\mathrm{d}x = \int_1^2 \left(2x^3 - \frac{1}{x} + 3x^{-2}\right) dx$

$$= \left(\cancel{2}\frac{x^4}{\cancel{4}\,2} - \ln|x| - \frac{3}{x}\right)\Bigg|_1^2$$

$$= 8 - \ln 2 - \frac{3}{2} - \left(\frac{1}{2} - 0 - 3\right)$$

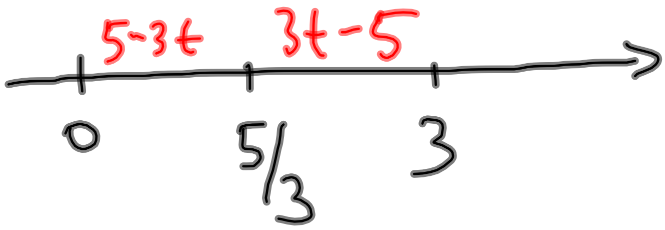
$$= 9 - \ln 2$$

2. $\int_0^3 |3t-5| \, dt$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

$$|3t-5| = \begin{cases} 3t-5, & 3t-5 \geq 0 \\ -(3t-5), & 3t-5 \leq 0 \end{cases}$$

$$= \begin{cases} 3t-5, & t \geq 5/3 \\ 5-3t, & \underline{t \leq 5/3} \end{cases}$$



$$\int_0^3 |3t-5| = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$$

$$= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3$$

$$= \dots = \boxed{\frac{41}{6}}$$

Applications of the Fundamental Theorem

If a particle is moving along a straight line then application of the Fundamental Theorem to $s'(t) = v(t)$ yields:

$$\int_{t_1}^{t_2} s'(t)\,dt = \boxed{\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1) = \text{displacement.}}$$

Show that

$$\underline{\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t.}$$

$s'(t) = v(t)$

$s(t)$ is antiderivative of $v(t)$

EXAMPLE 6. *A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled by the particle during the time period $1 \le t \le 6$.*

$$s'(t) = v(t) = t^2 - 2t - 8$$

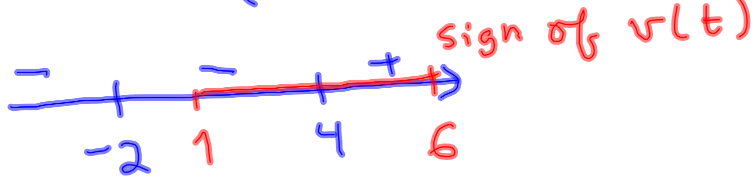
$$s(t) = \frac{t^3}{3} - t^2 - 8t + C$$

initial velocity

Total distance traveled =

$$\int_1^6 |v(t)|\,dt = \int_1^6 |t^2 - 2t - 8|\,dt$$

$$t^2 - 2t - 8 = (t+2)(t-4)$$



$$v(t) = \begin{cases} t^2 - 2t - 8, & 4 \le t \le 6 \\ 8 + 2t - t^2 & 1 \le t \le 4 \end{cases}$$

$$\int_1^6 |v(t)|\,dt = \int_1^4 (8 + 2t - t^2)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt$$

$$= \frac{98}{3}.$$