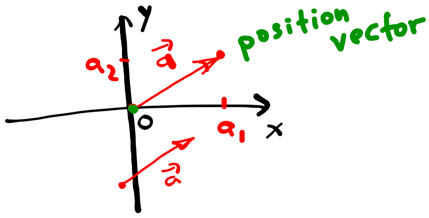
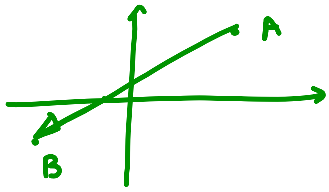
Section 1.1: Vectors

Quantities that we measure that have magnitude but not direction are called **scalars**.

DEFINITION 1. *A* ***vector*** *is a quantity that has both magnitude and direction. A 2-dimensional vector is an ordered pair* $\mathbf{a} = \langle a_1, a_2 \rangle$. *The numbers* a_1 *and* a_2 *are called the* ***components*** *of the vector* $\mathbf{a}$.





Typical notation to designate a vector is a boldfaced character or a character with and arrow on it (i.e. $\mathbf{a}$ or $\vec{a}$).

initial point end point

DEFINITION 2. *Given the points* $A(a_1, a_2)$ *and* $B(b_1, b_2)$, *the vector* $\mathbf{a}$ *with representation* $\overrightarrow{AB}$ *is*

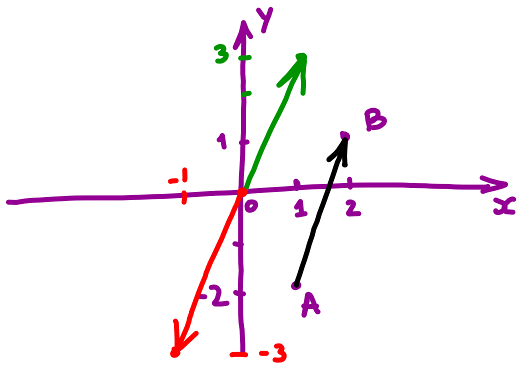
$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle .$$

The point A here is initial point and B is terminal one.

A vector with the initial point located at the origin is called the **position** vector (or we say that a vector is in standard position).

Vectors are equal if they have the same length and direction (same slope).

EXAMPLE 3. *Graph the vector with initial point A(1, -2) and terminal point B(2,1). Find the components of* $\overrightarrow{AB}$ *and* $\overrightarrow{BA}$.



$\overrightarrow{AB} = \langle 2-1, 1-(-2) \rangle = \langle 1, 3 \rangle$

$\overrightarrow{BA} = \langle 1-2, -2-1 \rangle = \langle -1, -3 \rangle$

$\boxed{\overrightarrow{AB} = -\overrightarrow{BA}}$

0

Vector operations

- *Scalar Multiplication*: If c is a scalar and $\mathbf{a} = \langle a_1, a_2 \rangle$, then

$$c\mathbf{a} = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle .$$

$5\langle 7,1\rangle = \langle 35,5\rangle$

$\langle 16,8\rangle = 8\langle 2,1\rangle$

DEFINITION 4. *Two vectors* **a** *and* **b** *are called* **parallel** *if* $\mathbf{b} = c\mathbf{a}$ *with some scalar* c.

If $c > 0$ then a and ca have the same direction, if $c < 0$ then a and ca have the opposite direction.

$(-2)\langle 1,1\rangle = \langle -2,-2\rangle \Rightarrow \langle 1,1\rangle \parallel \langle -2,-2\rangle$

$\vec{b} = c\vec{a} \Rightarrow \langle b_1, b_2\rangle = c\langle a_1, a_2\rangle$

$\langle b_1, b_2\rangle = \langle ca_1, ca_2\rangle$

$$\left.\begin{aligned} b_1 = ca_1 &\Rightarrow c = \frac{b_1}{a_1} \\ b_2 = ca_2 &\Rightarrow c = \frac{b_2}{a_2} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \dfrac{b_1}{a_1} = \dfrac{b_2}{a_2}$ proportional coordinates

$\langle 3,8\rangle \nparallel \langle -24,64\rangle$

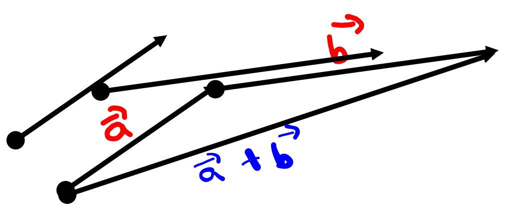
$\dfrac{3}{-24} \neq \dfrac{8}{64}$ $\qquad \langle 7,0\rangle \nparallel \langle 1,1\rangle$

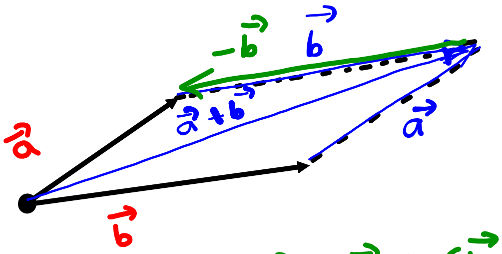
- *Vector addition*: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle .$$

$\langle 3, 4 \rangle + \langle -1, 0 \rangle = \langle 3 + (-1), 4 + 0 \rangle = \langle 2, 4 \rangle$

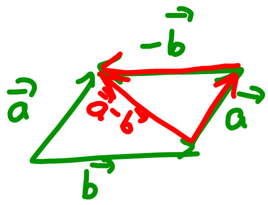
Triangle Law **Parallelogram Law**





$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$\mathbf{a} + \mathbf{b}$ is called the **resultant vector**



$\vec{b} - \vec{a} = -\vec{a} + \vec{b} = -(\vec{a} - \vec{b})$

EXAMPLE 5. *Let* $\mathbf{a} = \langle -1, 2 \rangle$ *and* $\mathbf{b} = \langle 2.1, -0.5 \rangle$. *Then* $3\mathbf{a} + 2\mathbf{b} =$

$3\vec{a} + 2b = 3\langle -1, 2 \rangle + 2\langle 2.1, -0.5 \rangle$

$= \langle -3, 6 \rangle + \langle 4.2, -1 \rangle$

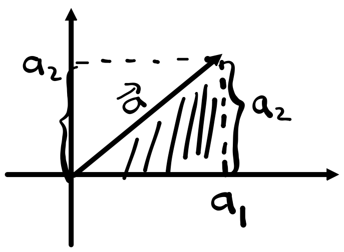
$= \langle -3 + 4.2, 6 + (-1) \rangle$

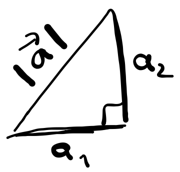
$= \langle 1.2, 5 \rangle$

Norm of a vector

The **magnitude**, the **norm**, or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $|\mathbf{a}|$,

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$





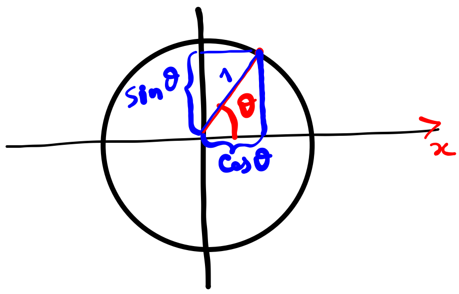
EXAMPLE 6. *Find:* $|\langle 3, -8 \rangle|$, $\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle\right|$, $|\mathbf{0}|$

$$|\langle 3, -8 \rangle| = \sqrt{3^2 + (-8)^2} = \sqrt{9+64} = \sqrt{73}$$

$$\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle\right| = 1$$

$$\left|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle\right| = 1$$

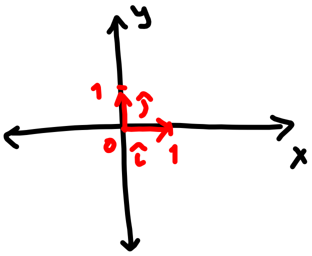
$$|\vec{0}| = |\langle 0, 0 \rangle| = 0$$



Unit vectors

A **unit** vector is a vector with length one. The **standard basis vectors** are given by the unit vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ along the x and y directions, respectively. Using the basis vectors, one can represent any vector $\mathbf{a} = \langle a_1, a_2\rangle$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$



$$\vec{a} = \langle a_1, a_2\rangle = \langle a_1, 0\rangle + \langle 0, a_2\rangle$$
$$= a_1\langle 1, 0\rangle + a_2\langle 0, 1\rangle$$
$$= a_1\hat{i} + a_2\hat{j}$$

EXAMPLE 7. *Given* $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \langle 5, -2\rangle$. *Find a scalars* s *and* t *such that* $s\mathbf{a} + t\mathbf{b} = -4\mathbf{j}$.

$$\parallel$$
$$\vec{a} = \langle 2, -1\rangle$$

Normalizing a vector

$|\hat{a}| = \left|\frac{\vec{a}}{|\vec{a}|}\right| = \left|\frac{1}{|\vec{a}|}\cdot\vec{a}\right| = \frac{1}{\cancel{|\vec{a}|}}\cancel{|\vec{a}|} = 1$

Any vector can be made into a unit vector by dividing it by its length. So, a unit vector in the direction of **a** is

$$\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.}$$

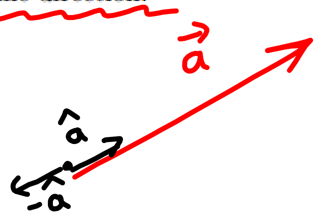
The process of multiplying a vector **a** by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing a**.

Any vector **a** can be fully represented by providing its length, $|\mathbf{a}|$ and a unit vector $\widehat{\mathbf{a}}$ in its direction:

$$\boxed{\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},}$$

i.e. *any vector is equal to its length times a unit vector in the same direction.*

EXAMPLE 8. *Given* $\mathbf{a} = \langle 2, -1\rangle$. *Find*



(a) *a unit vector that has the same direction as* **a**;

$|\vec{a}| = |\langle 2, -1\rangle| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, -1\rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$

(b) *a vector* **b** *in the direction opposite to* **a** *s.t* $|\mathbf{b}| = 7$.

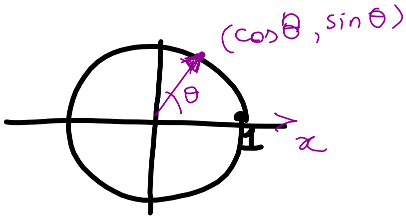
$\vec{b} = |\vec{b}|\cdot(-\hat{a}) = 7\cdot\left(-\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle\right)$

$= 7\left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = \left\langle -\frac{14}{\sqrt{5}}, \frac{7}{\sqrt{5}} \right\rangle$

Vectors determined by length and angle

If **a** is a nonzero position vector on the xy-plane that makes an angle θ with the positive x-axis then **a** can be expressed in trigonometric form as

$$\mathbf{a} = |\mathbf{a}| \underbrace{\langle \cos\theta, \sin\theta \rangle}_{\hat{a}}$$



For a unit vector this simplifies to

$$\boxed{\hat{\mathbf{a}} = \langle \cos\theta, \sin\theta \rangle}$$

EXAMPLE 9. *Find the vector* $\vec{a}$ *of length 5 that makes an angle* π*/4 with the positive* x*-axis.*

Given $|\vec{a}| = 5$, $\theta = \pi/4$

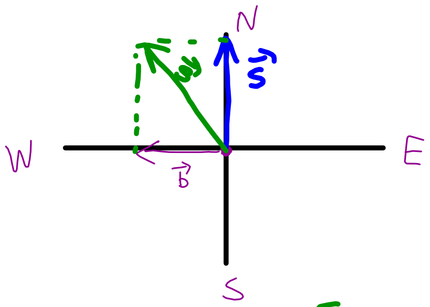
$$\vec{a} = |\vec{a}| \langle \cos\theta, \sin\theta \rangle = 5 \langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4} \rangle = 5 \left\langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2} \right\rangle$$

$$\boxed{\vec{a} = \left\langle \tfrac{5}{2}\sqrt{2}, \tfrac{5}{2}\sqrt{2} \right\rangle}$$

Application 1:

Quantities such as force, displacement or velocity that have direction as well as magnitude are represented by vectors.

EXAMPLE 10. *Ben walks due west on the deck of a ship at 5 mph* (speed). *The ship is moving north at a speed of 25 mph. Find the direction and speed of Ben relative to the surface of the water.*



$$\left.\begin{array}{l}\hat{b} = -\hat{\imath} = \langle -1, 0\rangle \\ |\vec{b}| = 5\end{array}\right\} \Rightarrow \vec{b} = 5\langle -1, 0\rangle = \langle -5, 0\rangle$$

$$\left.\begin{array}{l}\hat{s} = \hat{\jmath} = \langle 0, 1\rangle \\ |\vec{s}| = 25\end{array}\right\} \Rightarrow \vec{s} = 25\langle 0, 1\rangle = \langle 0, 25\rangle$$

Find the direction and magnitude of the resultant velocity

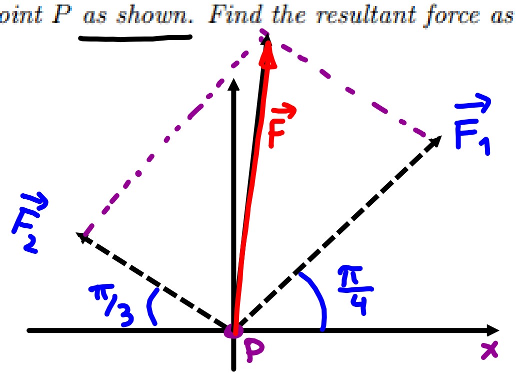
$$\vec{r} = \vec{b} + \vec{s} = \langle -5, 0\rangle + \langle 0, 25\rangle = \langle -5, 25\rangle = 5\langle -1, 5\rangle$$

speed $\quad |\vec{r}| = |5\langle -1, 5\rangle| = 5\,|\langle -1, 5\rangle| = 5\sqrt{(-1)^2 + 5^2} = \boxed{5\sqrt{26}}$

direction $\quad$ slope $\quad \tan\theta = \frac{5}{-1} = -5$

$$\boxed{\theta = \tan^{-1}(-5)}$$

EXAMPLE 11. *Two forces F_1 and F_2 with magnitudes 14 pounds and 12 pounds act on an object at a point P as shown. Find the resultant force as well as it's magnitude and direction.*



Given: $|\vec{F_1}| = 14\,lb$, $\theta_1 = \pi/4$

$|\vec{F_2}| = 12\,lb$, $\theta_2 = \frac{2\pi}{3}$

Find $\vec{F} = \vec{F_1} + \vec{F_2}$

$|\vec{F}|$, slope

$$\vec{F_1} = |\vec{F_1}|\langle \cos\theta_1, \sin\theta_1\rangle = 14\langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\rangle = 14\langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle$$

$$\vec{F_1} = \langle 7\sqrt{2}, 7\sqrt{2}\rangle$$

$$\vec{F_2} = 12\langle \cos\tfrac{2\pi}{3}, \sin\tfrac{2\pi}{3}\rangle = 12\langle -\cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3}\rangle = 12\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\rangle$$

$$\vec{F_2} = \langle -6, 6\sqrt{3}\rangle$$

$$\vec{F} = \vec{F_1} + \vec{F_2} = \langle 7\sqrt{2}, 7\sqrt{2}\rangle + \langle -6, 6\sqrt{3}\rangle = \langle \underbrace{7\sqrt{2}-6}, \underbrace{7\sqrt{2}+6\sqrt{3}}\rangle$$

$$\boxed{|\vec{F}| = \sqrt{(7\sqrt{2}-6)^2 + (7\sqrt{2}+6\sqrt{3})^2}}$$ magnitude

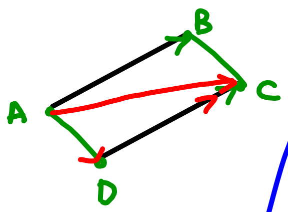
$$= \sqrt{49\cdot 2 - 2\cdot 7\cdot\sqrt{2}\cdot 6 + 36 + 49\cdot 2 + 2\cdot 7\cdot 6\sqrt{6} + 36\cdot 3} = \ldots$$

direction $\quad \theta = \tan^{-1}\left(\frac{7\sqrt{2}+6\sqrt{3}}{7\sqrt{2}-6}\right)$

Application 2:

Proofs of geometric facts via vector techniques.

EXAMPLE 12. *A quadrilateral has one pair of opposite sides parallel and of equal length. Use vectors to prove that the other pair of opposite sides is parallel and of equal length.* (i.e. this is a parallelogram)



Given: $\vec{AB} = \vec{DC}$

Prove: $\vec{DA} = \vec{CB}$, or $\vec{AD} = \vec{BC}$

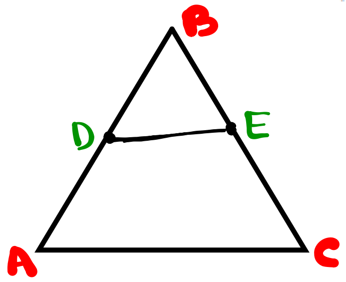
By the Triangle Law

$$\vec{AC} = \boxed{\vec{AD} + \vec{DC} = \vec{AB} + \vec{BC}}$$

$$-\vec{DC} \qquad -\vec{DC}$$

$$\star\ \vec{AD} = \underbrace{\vec{AB} - \vec{DC}}_{\vec{0}} + \vec{BC} = \vec{0} + \vec{BC} = \vec{BC}$$

EXAMPLE 13. *Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.*



Given $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$

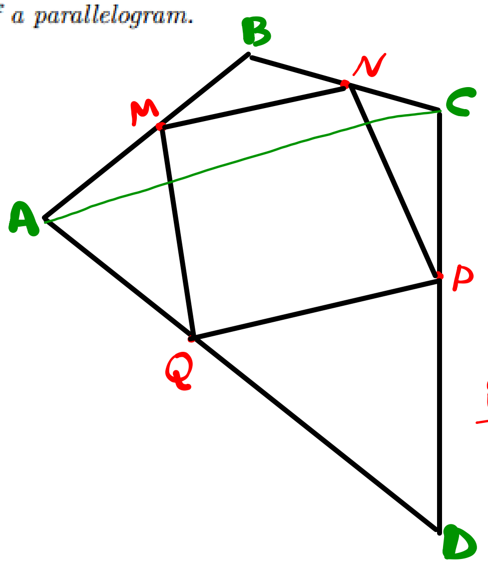
$\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$

Prove: $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$

Proof By the Triangle Law for ΔDBE:

$$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}AC$$

EXAMPLE 14. *Use vectors to prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.*



Given: $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$, or $\overrightarrow{BN} = \overrightarrow{NC}$

an so on....

Show By Ex. 12 it is sufficient to prove that

$\overrightarrow{MN} = \overrightarrow{QP}$

Proof Apply the Δ-le Law (See Ex. 13)

$$\left.\begin{array}{l} \Delta\, ABC \Rightarrow \overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC} \\ \Delta\, ADC \Rightarrow \overrightarrow{QP} = \frac{1}{2}\overrightarrow{AC} \end{array}\right\} \Rightarrow \overrightarrow{MN} = \overrightarrow{QP}$$