Section 1.2: The Dot Product

DEFINITION 1. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$ *is the number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

EXAMPLE 2. *Compute the dot product of* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

$\vec{a} \cdot \vec{b} = \langle 2, -3 \rangle \cdot \langle 3, -4 \rangle = 2 \cdot 3 + (-3) \cdot (-4) = 6 + 12 = 18$

$\vec{a} + \vec{b}$ vector

$c\vec{a}$ vector

$\vec{a} \cdot \vec{b}$ number

$\vec{a} \times \vec{b}$ in $\mathbb{R}^3$ vector

$\mathbb{R}^n$: $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$ $\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

Algebraic properties

THEOREM 3. If **a** $= \langle a_1, a_2 \rangle$, **b** $= \langle b_1, b_2 \rangle$, and **c** $= \langle c_1, c_2 \rangle$ are vectors and α ia a scalar, then

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) $\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot \alpha(\mathbf{b})$

$$5(\vec{a} \cdot \vec{b}) = (5\vec{a}) \cdot \vec{b} = \vec{a} \cdot (5\vec{b})$$

$$3(\langle \tfrac{1}{3}, \tfrac{1}{3} \rangle \langle 7, 6 \rangle) = \langle 1, 1 \rangle \cdot \langle 7, 6 \rangle = 7 + 6 = 13$$

(d) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(e) $\mathbf{0} \cdot \mathbf{a} = 0$

Proof.

(a)
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$
$$\underbrace{\vec{b} \cdot \vec{a}} = b_1 a_1 + b_2 a_2 = a_1 b_1 + a_2 b_2 = \underbrace{\vec{a} \cdot \vec{b}}$$

(b)
$$\underbrace{\vec{a} \cdot (\vec{b} + \vec{c})} = \langle \underbrace{a_1}, a_2 \rangle \cdot \langle \underbrace{b_1 + c_1}, b_2 + c_2 \rangle = a_1 (b_1 + c_1) + a_2 (b_2 + c_2)$$
$$= \underbrace{\underbrace{a_1 b_1} + \underbrace{a_2 b_2}}_{\vec{a} \cdot \vec{b}} + \underbrace{\underbrace{a_1 c_1} + \underbrace{a_2 c_2}}_{\vec{a} \cdot \vec{c}} = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

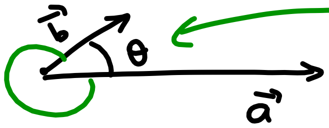
(c) prove by yourself

(d)
$$\vec{a} \cdot \vec{a} = \langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1 a_1 + a_2 a_2 = \overbrace{a_1^2 + a_2^2}^{\geq 0}$$
$$= \left(\sqrt{a_1^2 + a_2^2}\right)^2 = |\vec{a}|^2 \quad \Rightarrow \quad \vec{a} \cdot \vec{a} \geq 0$$

The property (d) of Theorem 3 implies a useful way of expressing the length of a vector in terms of dot product:

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a} \Longrightarrow |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

(e)
$$\vec{0} \cdot \vec{a} = 0$$
$$\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0 \cdot a_1 + 0 \cdot a_2 = 0$$



$\boxed{0 \le \theta \le \pi}$

THEOREM 4. *If* **a** *and* **b** *are two nonzero vectores and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{\vec a}{|\vec a|}\cdot\frac{\vec b}{|\vec b|} = \hat a \cdot \hat b \tag{1}$$

Note that the proof of the above theorem can be obtain by using the law of cosines and the algebraic properties of dot product.

$\cos\theta = -\frac{\sqrt{2}}{2}$ → $\theta = \frac{3\pi}{4} + 2\pi k$, $\theta = \frac{5\pi}{4} + 2\pi k$ $\Rightarrow$ Your answer: $\theta = \measuredangle\, \vec a, \vec b = \frac{3\pi}{4}$

Find angle Between $\langle 25, 10\rangle$ and $\langle -1000, 100\rangle$

$\langle 25,10\rangle = 5\langle 5, 2\rangle$ $\quad$ $\langle -1000, 100\rangle = 100\langle -10, 1\rangle$

Conclusion: This coincides with angle between $\langle 5,2\rangle$ and $\langle -10,1\rangle$

If $\cos\theta > 0$ $\Rightarrow$ θ is an acute angle

If $\cos\theta < 0$ $\Rightarrow$ θ is an obtuse angle

EXAMPLE 5. *Determine the angle between* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

$\vec{a} \cdot \vec{b} = \langle 2, -3 \rangle \cdot \langle 3, -4 \rangle = 2 \cdot 3 + (-3) \cdot (-4) = 6 + 12 = 18$

$|\vec{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$, $|\vec{b}| = \sqrt{3^2 + (-4)^2} = 5$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \dfrac{18}{5\sqrt{13}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{18}{5\sqrt{13}}\right)$

It will often be convenient to express (1) as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \tag{2}$$

where $0 \leq \theta \leq \pi$.

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely, $\theta = \frac{\pi}{2}$

THEOREM 6. *Two nonzero vectors* **a** *and* **b** *are orthogonal* ($\perp$) *if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Proof. Given $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$

We have to prove 2 statements:

(1) If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$

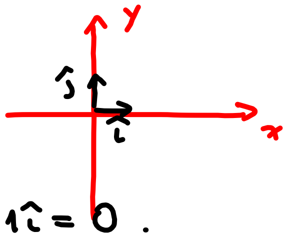
Proof: $\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \underline{\vec{a} \cdot \vec{b}} = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos\theta}_{\cos\frac{\pi}{2} = 0} = \underline{0}$

(2) If $\vec{a} \cdot \vec{b} = 0$ then $\vec{a} \perp \vec{b}$.

$\Downarrow$

$\underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cdot \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a} \perp \vec{b}$.

EXAMPLE 7. *What is the dot product of* $12\mathbf{j}$ *and* $11\mathbf{i}$?



$\hat{i} \perp \hat{j} \Rightarrow \angle\, \hat{i}, \hat{j} = \frac{\pi}{2}$

$\angle\, 12\hat{j},\ 11\hat{i} = \angle\, \hat{i}, \hat{j} = \frac{\pi}{2} \Rightarrow 12\hat{j} \cdot 11\hat{i} = 0$.

EXAMPLE 8. *Determine whether the given vectors are orthogonal, parallel, or neither. If the vectors are non orthogonal and non parallel, then determine whether the angle between them is acute or obtuse.*

(a) $\langle 3, 4\rangle$, $\langle -8, 6\rangle = 2\langle -4, 3\rangle$ Note

$\langle 3, 4\rangle \cdot \langle -8, 6\rangle = 3\cdot(-8) + 4\cdot 6 = 0 \Rightarrow \boxed{\langle 3, 4\rangle \perp \langle -8, 6\rangle}$

(b) $\langle -7, -4\rangle$, $\langle 28, 16\rangle = 4\langle 7, 4\rangle$ parallel

$\frac{-7}{28} = \frac{-4}{16} = -4$

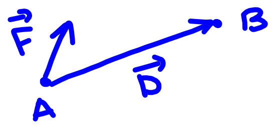
(c) $\langle -1, 1\rangle$, $\langle 2, -3\rangle$

$\langle -1, 1\rangle \cdot \langle 2, -3\rangle = -2 - 3 = -5$

$|\langle -1, 1\rangle| = \sqrt{2}$, $\quad |\langle 2, -3\rangle| = \sqrt{4+9} = \sqrt{13}$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-5}{\sqrt{2}\sqrt{13}} = -\frac{5}{\sqrt{26}}$

$\theta = \cos^{-1}\left(-\frac{5}{\sqrt{26}}\right)$ obtuse.



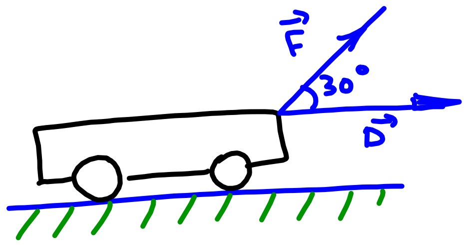
DEFINITION 9. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

Question: If you push against a wall, you may tire yourself out, but you will not perform any work. Why? $\vec{D} = \vec{0} \Rightarrow W = \vec{F} \cdot \vec{0} = 0$

EXAMPLE 10. *A wagon is pulled horizontally by exerting a force of* $50lb$ *on the handle at an angle* 30^o *with the horizontal. How much work is done in moving the wagon* $10ft$.



$|\vec{F}| = 50\ lb$

$\theta = \frac{\pi}{6}$

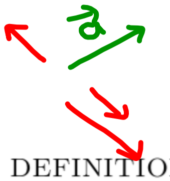
$|\vec{D}| = 10\ ft$

$$W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cdot \cos\theta = 50 \cdot 10 \cdot \cos\frac{\pi}{6}$$
$$= 500\,\frac{\sqrt{3}}{2} = 250\sqrt{3}\ ft\text{-}lb$$

EXAMPLE 11. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ $= \langle 25, 4\rangle$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1,1)$ *to* $B(5,6)$. *Find the work done if the distance is measured in meters.*

$$\vec{D} = \vec{AB} = \vec{OB} - \vec{OA} = \langle 5,6\rangle - \langle 1,1\rangle = \langle 5-1, 6-1\rangle = \langle 4,5\rangle$$

$$W = \vec{F}\cdot\vec{D} = \langle 25,4\rangle \cdot \langle 4,5\rangle = 25\cdot 4 + 4\cdot 5 = \boxed{120}\ J$$



$|\vec{a}^{\perp}| = \sqrt{(-a_2)^2 + a_1^2} = \sqrt{a_1^2 + a_2^2}$ $|\vec{a}|$

DEFINITION 12. *The* **orthogonal compliment of a** $= \langle a_1, a_2 \rangle$ **is** $\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle$.

Note that $|\mathbf{a}| = |\mathbf{a}^{\perp}|$ and $\mathbf{a} \cdot \mathbf{a}^{\perp} =$ $\langle a_1, a_2 \rangle \cdot \langle -a_2, a_1 \rangle = -a_1 a_2 + a_2 a_1 = 0 \Rightarrow \vec{a} \perp \vec{a}^{\perp}$

EXAMPLE 13. *Given* $\underbrace{\langle 4, -2 \rangle, \langle 2, -1 \rangle, \langle -2, 1 \rangle}_{\text{parallel}}$ *and* $\mathbf{a} = \langle 1, 2 \rangle$. *Which of these vectors is*

$\Rightarrow$ it is sufficient to verify if $\vec{a}$ is orthogonal to one of them.

- *orthogonal to* **a**?

$\langle 4, -2 \rangle \cdot \langle 1, 2 \rangle = 4 - 4 = 0 \Rightarrow \vec{a} \perp \langle 4, -2 \rangle \Rightarrow$ All 3 vectors are orthogonal to $\vec{a}$.

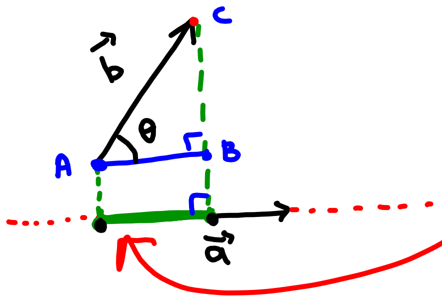
- *the orthogonal compliment of* **a**? $\vec{a}^{\perp} = \langle -2, 1 \rangle$.

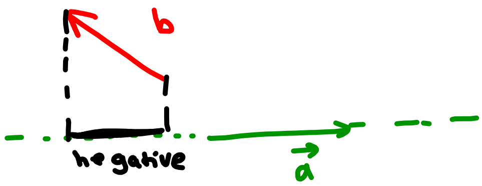
- The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is denotes by $\text{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (or the component of $\mathbf{b}$ along $\mathbf{a}$) is denotes by $\text{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \underbrace{\hat{a}\cdot\vec{b}}$$





$$\text{comp}_{\vec{a}}\vec{b} = |AB| = |AC|\cos\theta = |\vec{b}|\cos\theta = \cancel{|\vec{b}|}\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\cancel{|\vec{b}|}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\theta = \measuredangle\, \vec{b}, \vec{AB} = \measuredangle\, \vec{b}, \vec{a}$$

Remark If $c > 0$

$$\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{c\vec{a}}\vec{b}$$

$$\mathrm{proj}_{\vec{a}}\,\vec{b} = \underbrace{\mathrm{comp}_{\vec{a}}\,\vec{b} \cdot \hat{a}}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \cdot \hat{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$(\hat{a}\cdot\vec{b})\,\hat{a}$$

- The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is denotes by $\text{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (or the component of $\mathbf{b}$ along $\mathbf{a}$) is denotes by $\text{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

EXAMPLE 14. *Given* $\mathbf{a} = \langle 4, 3\rangle$ *and* $\mathbf{b} = \langle 1, -1\rangle$. *Find:*

- $\mathbf{a} \cdot \mathbf{b} =$ $\langle 4, 3\rangle \cdot \langle 1, -1\rangle = 4 - 3 = 1$

- $|\mathbf{a}| =$ $\sqrt{4^2 + 3^2} = 5$

- $|\mathbf{b}| =$ $\sqrt{1^2 + (-1)^2} = \sqrt{2}$

- $\operatorname{proj}_{\mathbf{b}} \mathbf{a} =$ $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \dfrac{1}{(\sqrt{2})^2} \langle 1, -1\rangle = \boxed{\langle \tfrac{1}{2}, -\tfrac{1}{2}\rangle}$

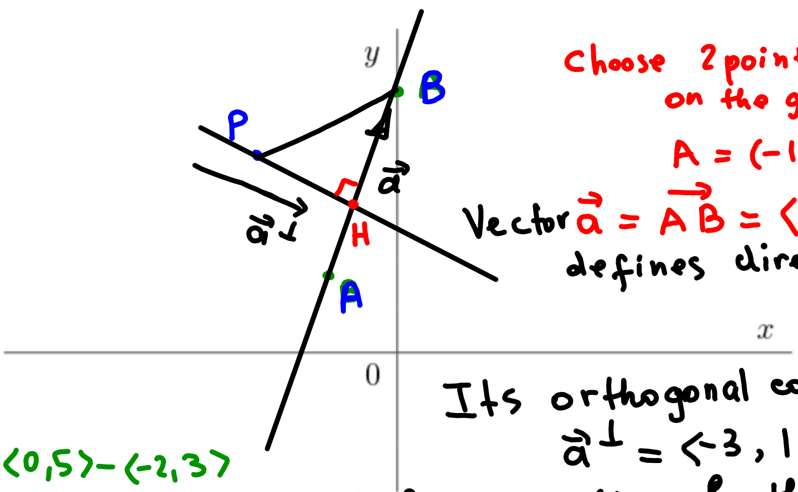
- $\operatorname{comp}_{\mathbf{a}} \mathbf{b} =$ $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \boxed{\dfrac{1}{5}}$

EXAMPLE 15. *Find the distance from the point $P(-2,3)$ to the line $y = 3x + 5$.*

x	-1	0
y	2	5
	A	B

Choose 2 points on the given line:

$A = (-1, 2)$, $B = (0, 5)$



Vector $\vec{a} = \overrightarrow{AB} = \langle 0-(-1), 5-2\rangle = \langle 1, 3\rangle$ defines direction of the given line.

Its orthogonal compliment $\vec{a}^{\perp} = \langle -3, 1\rangle$ defines direction of the line perpendicular to the given line.

$\overrightarrow{PB} = \langle 0,5\rangle - \langle -2,3\rangle$

$\overrightarrow{PB} = \langle 2,2\rangle$

It follows from $\triangle PBH$ that

$$|PH| = \left|\operatorname{comp}_{\vec{a}^{\perp}} \overrightarrow{PB}\right| = \left|\frac{\overrightarrow{PB}\cdot\vec{a}^{\perp}}{|a^{\perp}|}\right|$$

$$= \left|\frac{\langle 2,2\rangle \cdot \langle -3,1\rangle}{|\langle -3,1\rangle|}\right| = \left|\frac{-6+2}{\sqrt{9+1}}\right| = \boxed{\frac{4}{\sqrt{10}}}$$