Section 1.3: Vector functions

Parametric equations:

$$\boxed{x = x(t), \quad y = y(t)}$$

where the variable t is called a **parameter.** Each value of the parameter t defines a point that we can plot. As t varies over its domain we get a collection of points $(x, y) = (x(t), y(t))$ on the plane which is called the **parametric curve.**

Each parametric curve can be represented as the **vector function**:

$$\overrightarrow{r(t)} = \langle x(t), y(t) \rangle \, .$$

Note that Parametric curves have a **direction of motion** given by increasing of parameter t. So, when sketching parametric curves we also include arrows that show the direction of motion.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

parameter domain.

EXAMPLE 1. *Examine the parametric curve* $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 3\pi/2.$

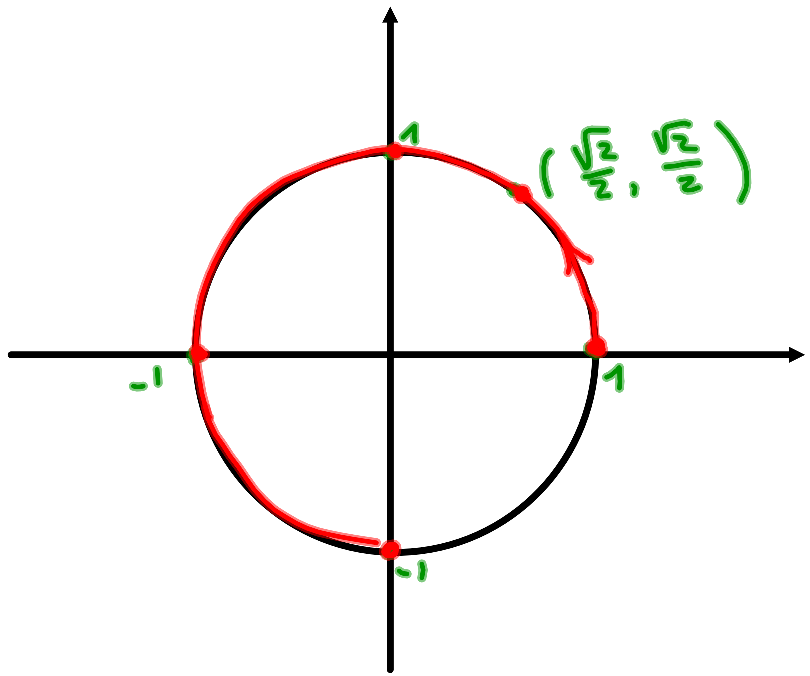
Eliminating the parameter t

$\cos^2 t + \sin^2 t = 1$

$\boxed{x^2 + y^2 = 1}$

unit circle centered at origin.

t	0	$\pi/4$	$\pi/2$	π	$\dots$	$3\pi/2$
(x,y)	$(1,0)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$(0,1)$	$(-1,0)$		$(0,-1)$



EXAMPLE 2. *Given* $\mathbf{r}(t) = \langle t+1, t^2 \rangle$.   (or $x(t) = t+1$, $y(t) = t^2$)

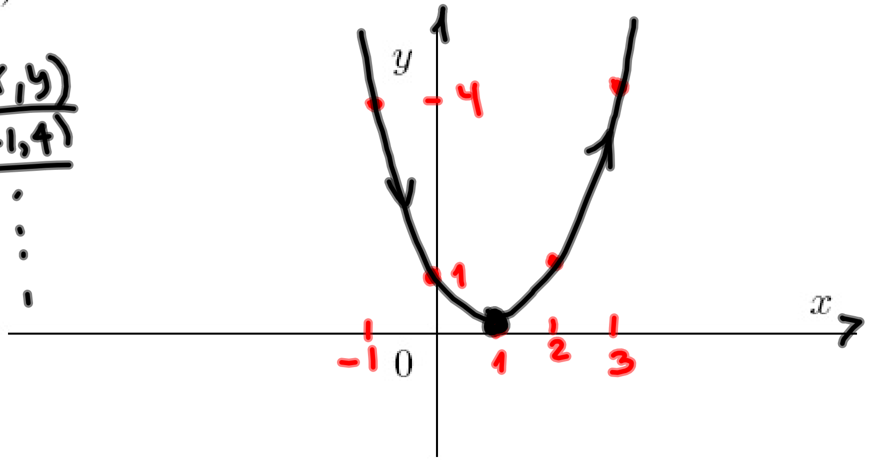
(a) *Does the point* $(4,3)$ *belong to the graph of* $\mathbf{r}(t)$? NO

$\vec{r}(t) = \langle 4,3 \rangle \Rightarrow \langle t+1, t^2 \rangle = \langle 4,3 \rangle$

$$\begin{cases} t+1 = 4 & \Rightarrow t = 3 \\ t^2 = 3 & \Rightarrow t = \pm\sqrt{3} \end{cases}$$ impos

(b) *Sketch the graph of* $\mathbf{r}(t)$.

t	$\mathbf{r}(t)$	(x,y)
-2	$\langle -1, 4 \rangle$	$(-1,4)$
-1	$\langle 0, 1 \rangle$	⋮
0	$\langle 1, 0 \rangle$	
1	$\langle 2, 1 \rangle$	
2	$\langle 3, 4 \rangle$	



(c) *Find the Cartesian equation of* $\mathbf{r}(t)$ *eliminating the parameter.*

$\vec{r}(t) = \langle t+1, t^2 \rangle \Rightarrow (x,y) = (t+1, t^2)$

$$\begin{cases} x = t+1 \\ y = t^2 \end{cases} \Rightarrow t = x-1 \qquad \boxed{y = (x-1)^2} \text{ parabola}$$

EXAMPLE 3. *Find the Cartesian equation for* $\mathbf{r}(t) = \underbrace{\cos t}_{x(t)}\mathbf{i} + \underbrace{\cos(2t)}_{y(t)}\mathbf{j}$

$x = \cos t$

$y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$

$\boxed{y = 2x^2 - 1}$ parabola

Appendix D

$\sin^2 x + \cos^2 x = 1$

$\sin 2x = 2 \sin x \cos x$

$\cos(2t) = \cos^2 t - \sin^2 t$

$\cos(2t) = 2\cos^2 t - 1$

$\cos(2t) = 1 - 2 \sin^2 t$

EXAMPLE 4. *An object is moving in the xy-plane and its position after t seconds is given by* $\mathbf{r}(t) = \langle 1+t^2, 1+3t\rangle$.

(a) *Find the position of the object at time $t = 0$.*

$$\vec{r}(0) = \langle 1+0^2, 1+3\cdot 0\rangle = \boxed{\langle 1, 1\rangle}$$

(b) *At what time does the object reach the point* $(10, 10)$. $\to \langle 10, 10\rangle$

$$\vec{r}(t) = \langle 1+t^2, 1+3t\rangle = \langle 10, 10\rangle$$

$$\begin{cases} 1+t^2 = 10 & \Rightarrow t^2 = 9 \Rightarrow t = \pm 3 \\ 1+3t = 10 & \Rightarrow 3t = 9 \Rightarrow t = 3 \end{cases} \Rightarrow \boxed{t = 3}$$

(c) *Does the object pass through the point* $(20, 20)$*?*

$$\left.\begin{array}{l} 1+t^2 = 20 \Rightarrow t = \pm\sqrt{19} \\ 1+3t = 20 \Rightarrow t = 19/3 \end{array}\right\} \text{impossible}$$

NO

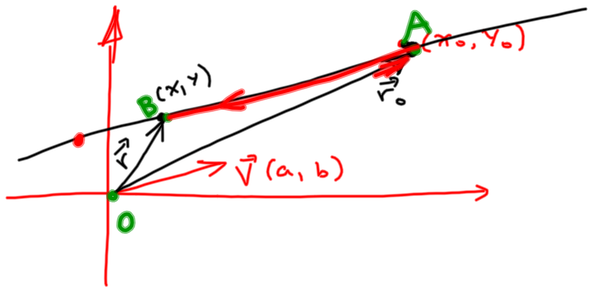
(d) *Find an equation in x and y whose graph is the path of the object.*

In other words, eliminate t.

$$\begin{cases} x = 1+t^2 \\ y = 1+3t \Rightarrow 3t = y-1 \Rightarrow t = \frac{y-1}{3} \end{cases}$$

$$x = 1 + \left(\frac{y-1}{3}\right)^2$$

$$\boxed{x = 1 + \frac{(y-1)^2}{9}}$$



$\vec r = \vec{OB} = \vec{OA} + \vec{AB}$

$\vec{OA} = \vec r_0$

$\vec{AB} = t\,\vec V$

A **Vector equation of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b \rangle$ is given by

$$\boxed{\mathbf{r} = \mathbf{r_0} + t\mathbf{v},}$$

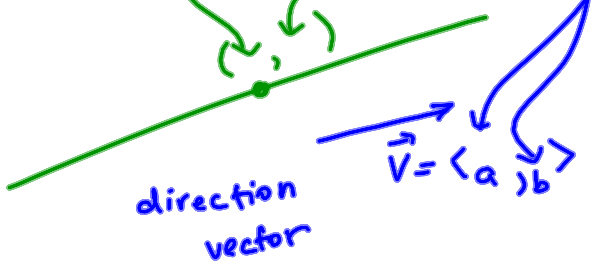
where $\mathbf{r_0} = \langle x_0, y_0 \rangle$.

The vector equation of the line can be rewritten in parametric form. Namely, we have

$$\begin{aligned} \langle x(t), y(t) \rangle &= \mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \\ &= \langle x_0, y_0 \rangle + t\,\langle a, b \rangle = \langle x_0, y_0 \rangle + \langle ta, tb \rangle = \\ &= \langle x_0 + ta, y_0 + tb \rangle\,. \end{aligned}$$

This immediately yields that the **parametric equations of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b \rangle$ are

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt.$$



EXAMPLE 5. *Find parametric equations of the line*

(a) *passing through the point* $(1,0)$ *and parallel to the vector* $\underbrace{\mathbf{i}-4\mathbf{j}}$;

$(x_0, y_0) = (1, 0)$

$\vec{V} = \langle 1, -4 \rangle$

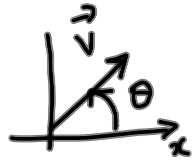
$$x = \underbrace{1}_{\vec{r}_0} + t\cdot \underbrace{1}_{\vec{V}}$$
$$y = 0 + t\cdot(-4)$$

$$\boxed{\begin{aligned} x &= 1+t \\ y &= -4t \end{aligned}}$$

Vector equation: $\vec{r}(t) = \vec{r}_0 + t\vec{V} \Rightarrow \vec{r}(t) = \langle 1+t, -4t \rangle$

(b) *passing through the point* $(-4,5)$ *with slope* $\sqrt{3}$;

$\boxed{(x_0, y_0) = (-4, 5)}$



$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\vec{V} = \langle \cos\theta, \sin\theta \rangle$

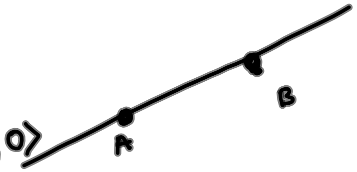
$\vec{V} = \langle \cos\pi/3, \sin\pi/3 \rangle$

$\boxed{\vec{V} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle}$

$$\boxed{\begin{aligned} x &= -4 + \tfrac{1}{2}t \\ y &= 5 + \tfrac{\sqrt{3}}{2}t \end{aligned}}$$

(c) *passing through the points* $\overset{A}{(7,2)}$ *and* $\overset{B}{(3,2)}$.

$(x_0, y_0) = (7, 2)$



$\vec{V} = \vec{AB} = \langle 3,2 \rangle - \langle 7,2 \rangle = \langle -4, 0 \rangle$

$$\boxed{\begin{aligned} x &= 7 - 4t \\ y &= 2 \end{aligned}}$$

$$\boxed{\begin{aligned} x &= 3 + 4t \\ y &= 2 \end{aligned}}$$

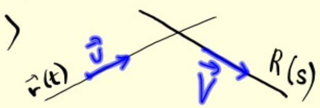
different parametrization

$$\boxed{\begin{aligned} x &= 7 + t \\ y &= 2 \end{aligned}}$$

EXAMPLE 6. *Determine whether the lines* $\mathbf{r}(t) = \langle 1+t, 1-3t\rangle$, $\mathbf{R}(s) = \langle 1+3s, 12+s\rangle$ *are parallel, orthogonal or neither. If they are not parallel, find the intersection point.*

$\vec{r}(t) = \langle 1+t, 1-3t\rangle$ $\quad$ $R(s) = \langle 1+3s, 12+s\rangle$

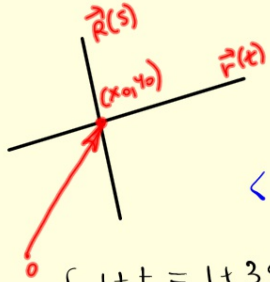
$\vec{v} = \langle 1, -3\rangle$ $\quad$ $\vec{V} = \langle 3, 1\rangle$



$\vec{v} \neq c\vec{V}$ or $\frac{1}{3} \neq \frac{-3}{1}$ $\Rightarrow \vec{v} \not\parallel \vec{V} \Rightarrow$ the lines are not parallel

Note that $\vec{v} = \vec{V}^{\perp}$ or $\vec{v}\cdot\vec{V} = \langle 1,-3\rangle\cdot\langle 3,1\rangle = 3-3=0$ $\rightarrow$ $\vec{v} \perp \vec{V} \Rightarrow$ the lines are orthogonal

Find intersection point



There are values of s and t such that

$$\vec{r}(t) = \vec{R}(s) = \boxed{\langle x_0, y_0\rangle}$$

$$\langle 1+t, 1-3t\rangle = \langle 1+3s, 12+s\rangle$$

$$\begin{cases} 1+t = 1+3s \Rightarrow t = 3s \\ 1-3t = 12+s \Rightarrow s+3t = -11 \end{cases}$$

$s + 3\cdot 3s = -11$

$10s = -11$

$s = -1.1$

$t = 3s = -3.3$

$\vec{r}(-3.3) = R(-1.1) = \langle 1-3.3, 1+3\cdot 3.3\rangle =$

$= \langle -2.3, 10.9\rangle$

$\boxed{(-2.3, 10.9)}$