Section 2.2: The Limit of a function

A limit is a way to discuss how the values of a function $f(x)$ behave when x approaches a number a, whether or not $f(a)$ is defined.

Let's consider the following function:

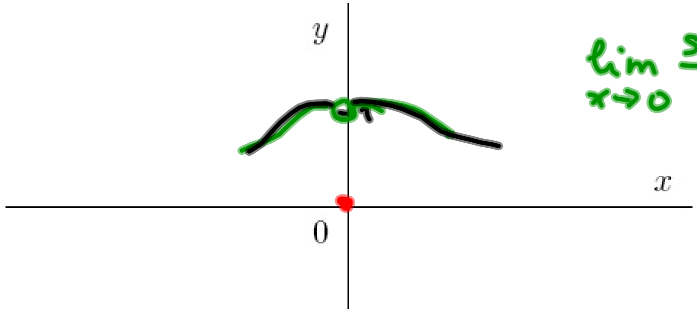
$$f(x) = \frac{\sin x}{x} \qquad (x \text{ in radians}).$$

$x = 0$ is not in domain of $f(x)$

Note that $f(0) = \dfrac{\sin 0}{0}$ is undefined. However, one can compute the values of $f(x)$ for values of x close to 0.

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = f(x) \quad \text{even}$$

x	$f(x)$
$\pm$ 0.1	0.99833417
$\pm$ 0.05	0.99958339
$\pm$ 0.01	0.99998333
$\pm$ 0.005	0.99999583
$\pm$ 0.001	0.99999983



$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

The table allows us to guess (correctly) that that our function gets closer and closer to 1 as x approaches 0 through positive and negative values. In limit notation it can be written as

$$\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

which implies that

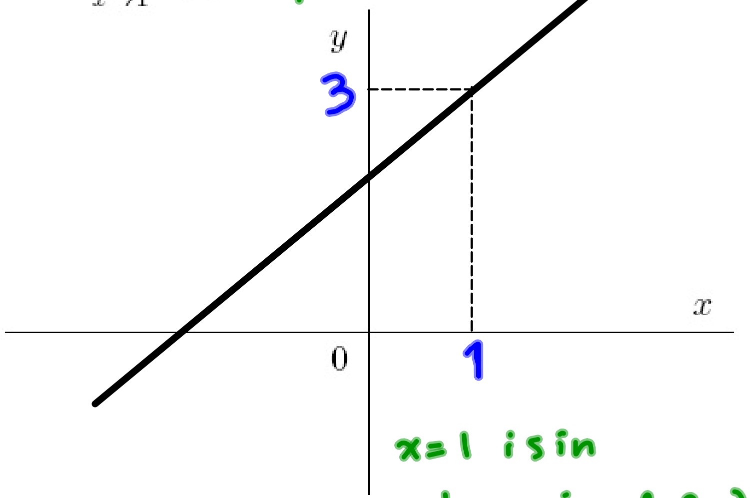
$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$$

FACT

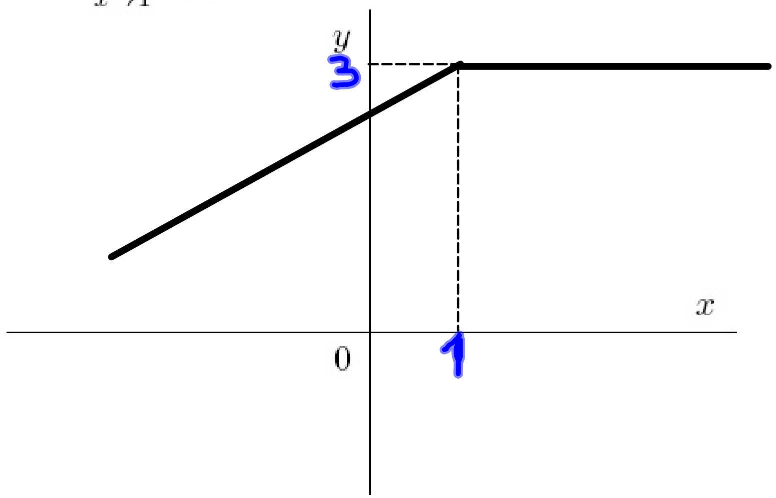
DEFINITION 1.

- *If* $\overset{\text{Left}}{\lim\limits_{x \to a^-}} f(x) = \overset{\text{Right}}{\lim\limits_{x \to a^+}} f(x) = L$ *then* $\lim\limits_{x \to a} f(x)$ *exists and* $\lim\limits_{x \to a} f(x) = L$;
- *If* $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x) = L$ *then* $\lim\limits_{x \to a} f(x)$ *does not exist.*

$\lim_{x\to 1} f(x) = f(1) = 3$



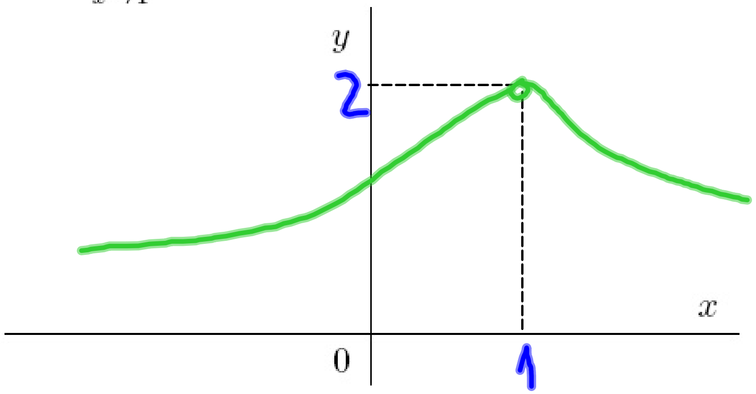
$\lim_{x\to 1} f(x) = f(1) = 3$



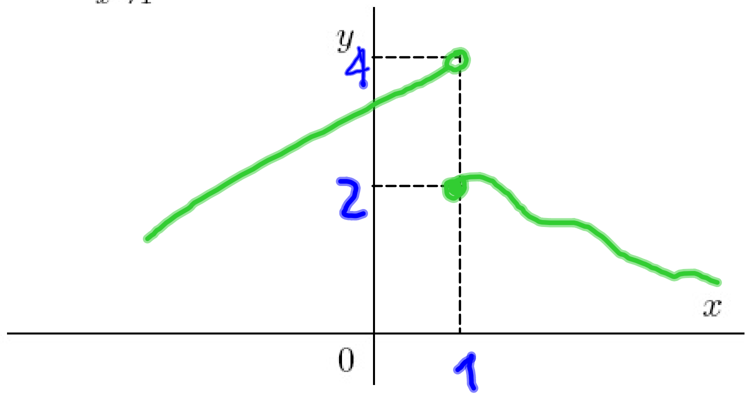
$x=1$ is in domain of $f(x)$

$\Rightarrow$ use "direct substitution" rule

$\lim_{x \to 1} f(x) = 2$



$\lim_{x \to 1} f(x) = \text{DNE}$



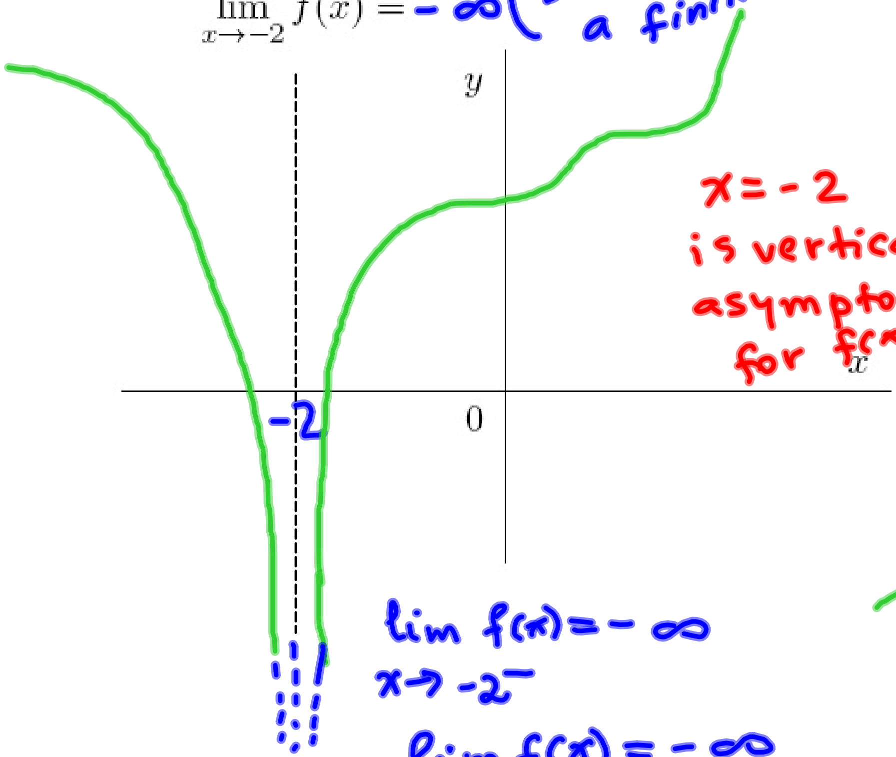
$x = 1$ is not in domain of f

Use left and right hand limits

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$$

$$\lim_{x \to 1^-} f(x) = 4 \neq \lim_{x \to 1^+} f(x) = 2$$

$$\lim_{x \to -2} f(x) = -\infty \text{ (DNE as a finite limit)}$$

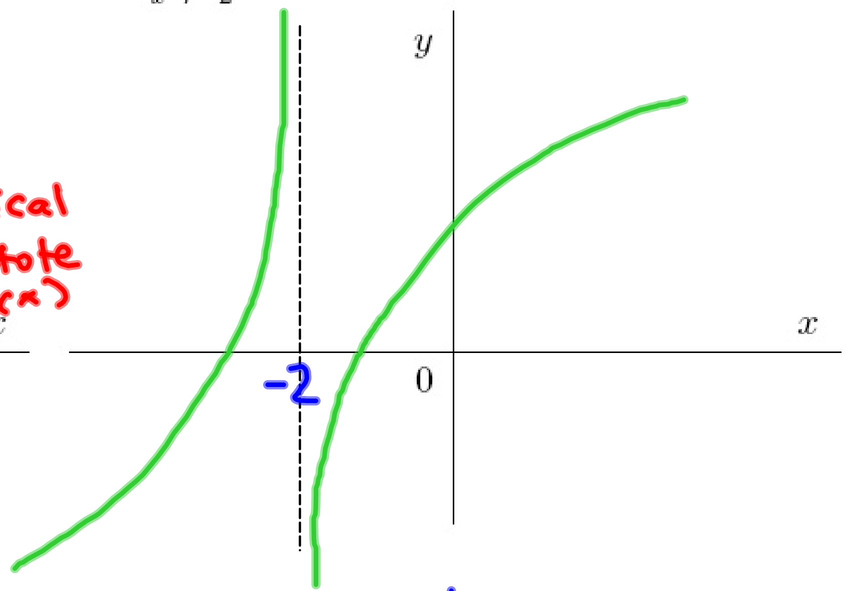


$x = -2$ is vertical asymptote for $f(x)$

$$\lim_{x \to -2^-} f(x) = -\infty$$

$$\lim_{x \to -2^+} f(x) = -\infty$$

$$\lim_{x \to -2} f(x) = \text{DNE}$$



$$\lim_{x \to -2^-} f(x) = +\infty$$

$\neq$

$$\lim_{x \to -2^+} f(x) = -\infty$$

Limits of piecewise defined function.

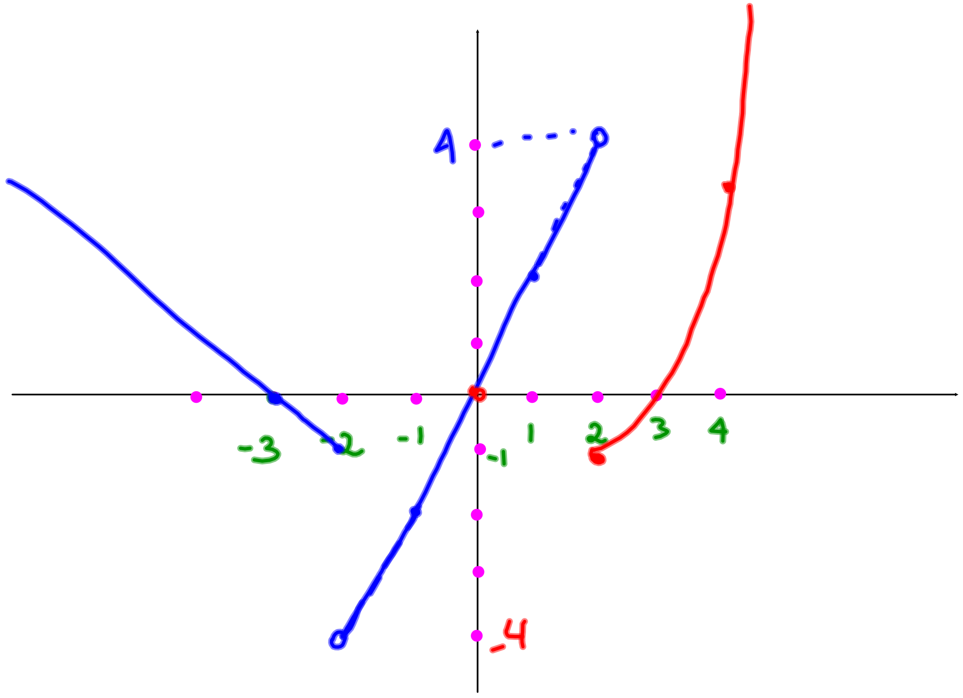
EXAMPLE 2. *Plot the graph of the function*

$$f(x) = \begin{cases} -3 - x & \text{if} \quad x \le -2 \\ 2x & \text{if} \quad -2 < x < 2 \\ x^2 - 4x + 3 & \text{if} \quad x \ge 2 \end{cases}$$

x	y
-3	0
-2	-1

x	y
-1	-2
1	2

x	y
2	~~3~~ -1
4	3



Find the limits (using the graph above):

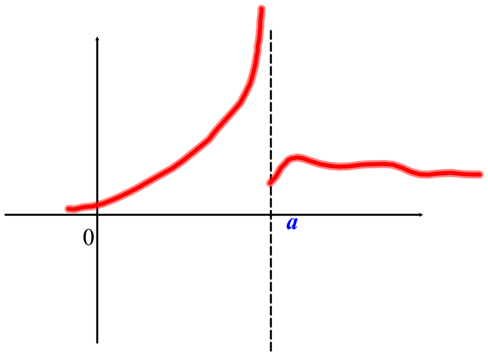
$\lim_{x\to 0^-} f(x) = 0$ $\qquad \lim_{x\to 0^+} f(x) = 0$ $\qquad \lim_{x\to 0} f(x) = 0$

$\lim_{x\to -2^-} f(x) = -1$ $\qquad \lim_{x\to -2^+} f(x) = -4$ $\qquad \lim_{x\to -2} f(x) =$ DNE

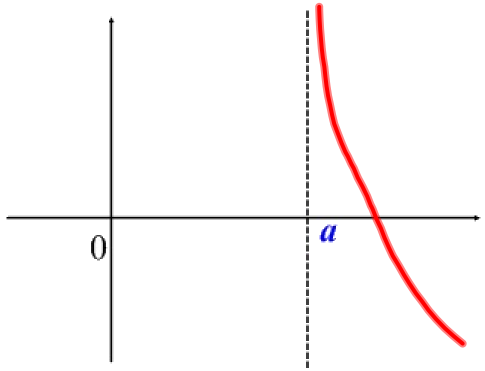
$\lim_{x\to 2^-} f(x) = 4$ $\qquad \lim_{x\to 2^+} f(x) = -1$ $\qquad \lim_{x\to 2} f(x) =$ DNE

DEFINITION 3. *The line $x = a$ is said to be a* **vertical asymptote** *of the curve $y = f(x)$ if at least one of the following six statements is true:*

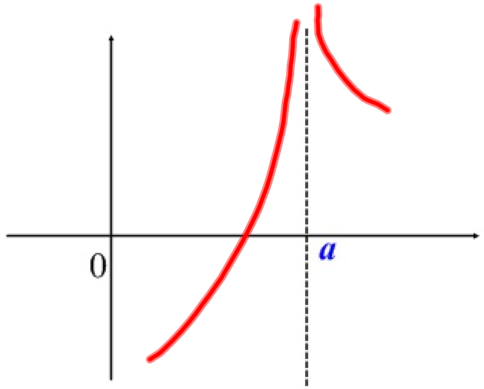
$$\lim_{x \to a^-} f(x) = \infty$$



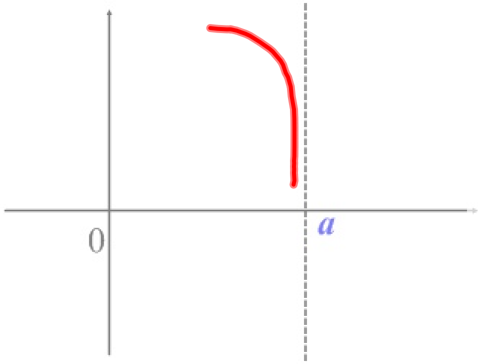
$$\lim_{x \to a^+} f(x) = \infty$$



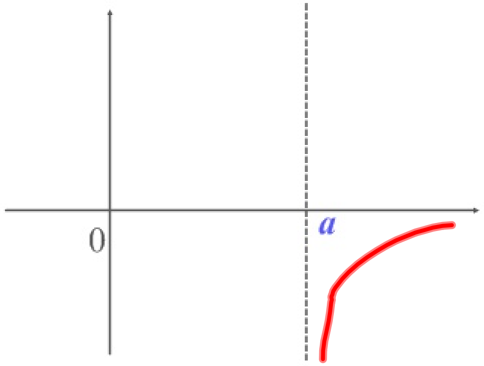
$$\lim_{x \to a} f(x) = \infty$$



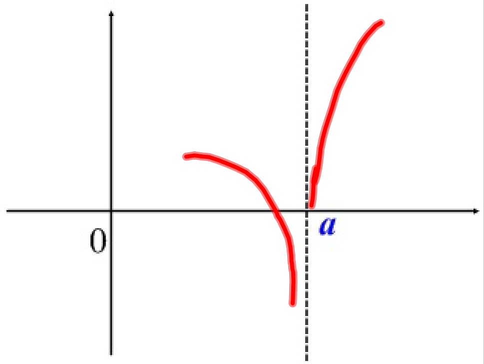
$$\lim_{x \to a^-} f(x) = -\infty$$



$$\lim_{x \to a^+} f(x) = -\infty$$



$$\lim_{x \to a} f(x) = -\infty$$



REMARK 4. The vertical asymptotes of a $\underbrace{\text{rational}}_{}$ function come from the zeroes of the denominator.

$\dfrac{\text{polynomial}}{\text{polynomial}}$

EXAMPLE 5. *Determine the infinite limit:*

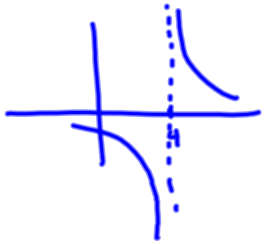
(a) $\lim_{x\to 4^-} \dfrac{7}{x-4} = -\infty$

rational

If $x > 4 \Rightarrow \dfrac{7}{x-4} > 0$

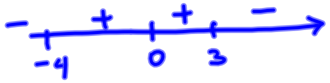
If $x < 4 \Rightarrow \dfrac{7}{x-4} < 0$

(b) $\lim_{x\to 4^+} \dfrac{7}{x-4} = +\infty$

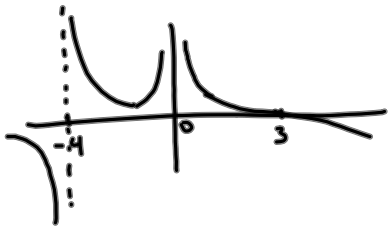


(c) $\lim_{x\to 4} \dfrac{7}{x-4} =$ DNE

(d) $\lim_{x\to 0^-} \dfrac{3-x}{x^4(x+4)} = \infty$



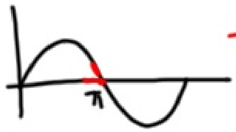
(e) $\lim_{x\to 0^+} \dfrac{3-x}{x^4(x+4)} = \infty$



(f) $\lim_{x\to 0} \dfrac{3-x}{x^4(x+4)} = \infty$

(g) $\lim_{x\to\pi^-} \csc x = \lim_{x\to\pi^-} \dfrac{1}{\sin x} = \infty$

Determine sign of $y = \sin x$ when $\underline{x < \pi}$ (closed to π)



$\rightarrow \sin x > 0 \Rightarrow \dfrac{1}{\sin x} > 0$

EXAMPLE 6. *Given:* $f(x) = \dfrac{x-4}{x^2-5x+4}$. rational

(a) *What are the vertical asymptotes of* $f(x)$? $x^2-5x+4=0$

Zeroes of the denom.

$x=1$, $x=4$. In other words, $x^2-5x+4=(x-1)(x-4)$

$$f(x) = \frac{\cancel{x-4}}{(x-1)\cancel{(x-4)}} = \frac{1}{x-1}, \quad x \neq 4$$

$\left.\begin{matrix} x=4 \\ x=1 \end{matrix}\right\}$ are not in the domain of f.

$x=1$ is vertical asymptote.

(b) *How does* $f(x)$ *behave near the asymptotes?*

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x-1} \quad \boxed{\text{DNE}}$$

$$\left.\begin{matrix} \dfrac{1}{x-1} > 0 \text{ if } x>1 \Rightarrow \lim\limits_{x\to 1^+} f(x) = +\infty \\ \neq \\ \dfrac{1}{x-1} < 0 \text{ if } x<1 \Rightarrow \lim\limits_{x\to 1^-} f(x) = -\infty \end{matrix}\right\}$$