Section 2.3: Calculating limits using the limits laws

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} g(x) = L_2$$

exist. Then

1. $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ $= L_1 \pm L_2$
2. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ $\lim_{x\to a} 3f(x) = 3\lim_{x\to a} f(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
4. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$

$\lim_{x\to a}(Af+Bg) = A\lim_{x\to a} f + B\lim_{x\to a} g$

6. $\lim_{x\to a} x = a$
7. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer. $\lim_{x\to 3}(\sin x)^4 = \left(\lim_{x\to 3}\sin x\right)^4$
8. $\lim_{x\to a} x^n = a^n$, where n is a positive integer.
9. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer and if n is even, then we assume that $\lim_{x\to a} f(x) > 0$.
10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}$ where n is a positive integer and if n is even, then we assume that $a > 0$.

$= \sqrt[n]{a}$

REMARK 1. Note that *all these properties also hold for the one-sided limits.*

REMARK 2. The analogues of the *laws 1-3 also hold when f and g are vector functions* (the product in Law 3 should be interpreted as a dot product).

$\vec{r}_1(t) = \langle 5-4t,\ 7t^3+2013\rangle$

$\vec{r}_2(t) = \langle 4t,\ 2-7t^3+t\rangle$

$\lim_{t\to 10}\vec{r}_1(t) + \lim_{t\to 10}\vec{r}_2(t) = \lim_{t\to 10}(\vec{r}_1+\vec{r}_2)$

$= \lim_{t\to 10}\langle 5-\cancel{4t}+\cancel{4t},\ \cancel{7t^3}+2013+2-\cancel{7t^3}+t\rangle$

$= \lim_{t\to 10}\langle 5,\ 2015+t\rangle = \langle 5, 2015+10\rangle = \langle 5, 2025\rangle$

EXAMPLE 3. *Compute the limit:*

$$\lim_{x\to -1}(7x^3-5) \overset{1}{=} \lim_{x\to -1} 7x^3 - \lim_{x\to -1} 5$$

$$\underline{\underline{2, 5}} \quad 7\lim_{x\to -1} x^3 - 5 \overset{8}{=} \underbrace{7\cdot(-1)^3-5} = -12$$

REMARK 4. If we had defined $f(x)=7x^3-5$ then Example 3 would have been,

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}(7x^3-5) = \underbrace{7(-1)^3-5} = -12 = f(-1)$$

$x=-1$ belongs to the domain of f.

DIRECT SUBSTITUTION PROPERTY

EXAMPLE 5. *Compute the limit:*

$$\lim_{x\to -2}\frac{x^2+x+1}{x^3-10} =$$

REMARK 6. The function from Example 5 also satisfies "direct substitution property":

$$\lim_{x\to a} f(x) = f(a).$$

Later we will say that such functions are *continuous*. Note that in both examples it was important that a in the domain of f.

EXAMPLE 7. *Compute the limit:*

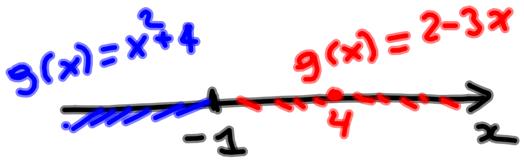
$$\lim_{x\to 3}\frac{x-3}{x^2-9} \overset{\text{FACTOR}}{=} \lim_{x\to 3}\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)} = \lim_{x\to 3}\frac{1}{x+3} \overset{\text{D.S.P.}}{=} \frac{1}{3+3} = \frac{1}{6}$$

EXAMPLE 8. *Compute the limit:*

$$\lim_{x\to 1}\frac{x-1}{x^2-4x+3} \overset{\text{Factor}}{=} \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x-3)} = \lim_{x\to 1}\frac{1}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$$

EXAMPLE 9. *Given*

$$g(x) = \begin{cases} x^2+4, & \text{if } x \le -1 \\ 2-3x & \text{if } x > -1 \end{cases}$$



Compute the limits:

(a) $\lim_{x\to 4} g(x) = \lim_{x\to 4} (2-3x) = 2-3\cdot 4 = 2-12 = -10$

(b) $\lim_{x\to -1} g(x) = 5$

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} (x^2+4) = (-1)^2+4 = 5$$

$$\|$$

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} (2-3x) = 2-3\cdot(-1) = 5$$

EXAMPLE 10. *Evaluate these limits.*

(a) $\lim_{x\to 4} \frac{x^{-1} - 0.25}{x-4} = \lim_{x\to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x-4} = \lim_{x\to 4} \frac{\frac{4-x}{4x}}{x-4}$

$= \lim_{x\to 4} \frac{-\cancel{(x-4)}}{4x\cancel{(x-4)}} = -\frac{1}{4}\lim_{x\to 4}\frac{1}{x} = -\frac{1}{4\cdot 4} = \boxed{-\frac{1}{16}}$

(b) $\lim_{x\to 0} \frac{(x+5)^2 - 25}{x} = \lim_{x\to 0} \frac{(x+5)^2 - 5^2}{x} = \lim_{x\to 0} \frac{(x+\cancel{5}-\cancel{5})(x+5+5)}{x}$

$= \lim_{x\to 0} \frac{\cancel{x}(x+10)}{\cancel{x}} = \lim_{x\to 0}(x+10) = 0+10 = \boxed{10}$

(c) $\lim_{x\to 0^-}\left\{\frac{1}{x}-\frac{1}{|x|}\right\} = \lim_{x\to 0^-}\frac{1}{2x} = -\infty$

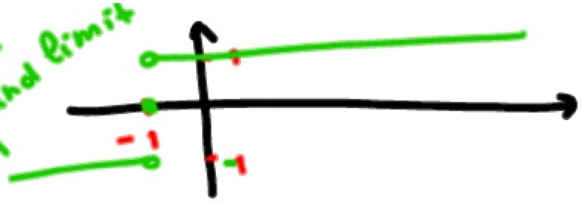
Left

$$|x| = \begin{cases} x, & x>0 \\ 0, & x=0 \\ -x, & x<0 \end{cases} \Rightarrow \frac{1}{|x|} = \begin{cases} \frac{1}{x}, & x>0 \\ -\frac{1}{x}, & x<0 \end{cases}$$

$$\frac{1}{x}-\frac{1}{|x|} = \begin{cases} \frac{1}{x}-\frac{1}{x}, & x>0 \\ \frac{1}{x}-\left(-\frac{1}{x}\right), & x<0 \end{cases} = \begin{cases} 0, & x>0 \\ \frac{1}{2x}, & x<0 \end{cases}$$

(d) $\lim_{x\to -1}\frac{|x+1|}{x+1}$

(d) $\lim\limits_{x\to -1} \dfrac{|x+1|}{x+1}$ DNE because left hand limit $\neq$ right hand limit at $x=-1$



$$\frac{|x+1|}{x+1} = \begin{cases} \frac{x+1}{x+1} = 1 & \text{if } x > -1 \\ \frac{0}{x+1} = 0 & \text{if } x = -1 \\ \frac{-(x+1)}{x+1} = -1 & \text{if } x < -1 \end{cases} = \begin{cases} 1 & \text{if } x > -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } x < -1 \end{cases}$$

(e) $\lim\limits_{x\to 0} \dfrac{\sqrt{6-x}-\sqrt{6}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{6-x}-\sqrt{6}}{x} \cdot \dfrac{\sqrt{6-x}+\sqrt{6}}{\sqrt{6-x}+\sqrt{6}}$

$$= \lim_{x\to 0} \frac{\cancel{6}-x-\cancel{6}}{x(\sqrt{6-x}+\sqrt{6})} = -\lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{6-x}+\sqrt{6})}$$

$$= -\lim_{x\to 0} \frac{1}{\sqrt{6-x}+\sqrt{6}} = -\frac{1}{\sqrt{6-0}+\sqrt{6}} = -\frac{1}{2\sqrt{6}}$$

Conclusion from the above examples:

`To calculate the limit of` $f(x)$ `as` $x \to a$:

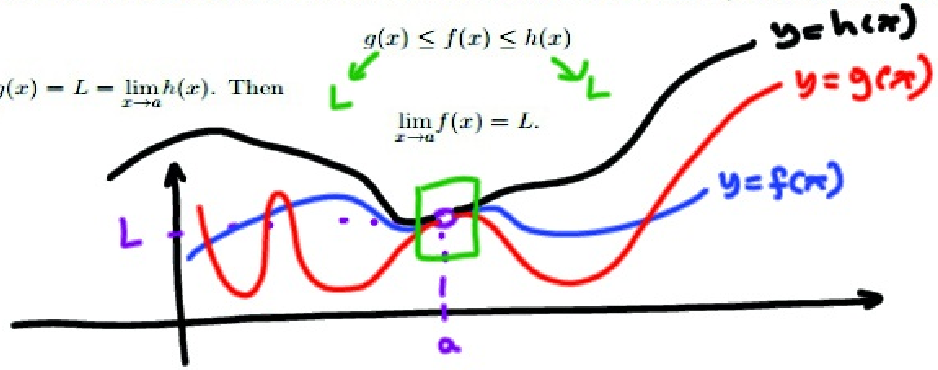
* `PLUG IN` $x = a$ `if` a `is in the domain of` f. (DIRECT SUBSTITUTION $\lim_{x\to a} f(x) = f(a)$)
* `Otherwise "FACTOR" or "MULTIPLY BY CONJUGATE" and then plug in.` (If $x=a$ is a common zero of both numerator and denominator)
* `Consider one sided limits if necessary.`

* **Squeeze Theorem.** Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$\underbrace{g(x)}_{L} \le f(x) \le \underbrace{h(x)}_{L}$$

and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then

$$\lim_{x\to a} f(x) = L.$$



Corollary. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$|f(x)| \le h(x) \quad (\textit{equivalently}, \quad -h(x) \le f(x) \le h(x))$$

and $\lim_{x\to a} h(x) = 0$. Then

$$\lim_{x\to a} f(x) = 0.$$

EXAMPLE 11. *Given* $\underbrace{3x}_{\downarrow\ 3} \le f(x) \le \underbrace{x^3 + 2}_{\downarrow x\to 1\ \ 3}$ *for* $0 \le x \le 2$. *Find* $\lim_{x\to 1} f(x)$

$$f(x) \xrightarrow[x\to 1]{} 3$$

EXAMPLE 12. *Evaluate:*

(a) $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

$-1 \leq \sin\frac{1}{x} \leq 1$ (Mult. by x)

$$\underbrace{-x}_{\substack{\downarrow x\to 0\\ 0}} \leq \underbrace{x\sin\frac{1}{x}}_{\substack{\downarrow \text{By Squeeze theorem}\\ 0}} \leq \underbrace{x}_{\substack{\downarrow x\to 0\\ 0}}$$

(b) $\lim_{t\to 0}(t^5)\cos^3(\frac{1}{t^2}) = 0$

$(-1)^3 \leq \cos^3\frac{1}{t^2} \leq 1^3$

$-1\cdot t^5 \leq t^5\cos^3\frac{1}{t^2} \leq 1\cdot t^5$

$$\underbrace{-t^5}_{\substack{\downarrow\\ 0}} \leq \underbrace{t^5\cos^3\frac{1}{t^2}}_{\substack{\downarrow \text{By squeeze theorem}\\ 0}} \leq \underbrace{t^5}_{\substack{\downarrow\\ 0}}$$

Def A Function $f(x)$ is called **bounded** on an open interval $I = (a, b)$ if there exists a number M such that

$$|f(x)| \le M \quad \text{for all } x \text{ in } I$$

$$(-M \le f(x) \le M)$$

* Conclusion: If $f(x)$ is bounded in a neighborhood of point $x = a$ and $\lim_{x \to a} g(x) = 0$ then

By Squeeze Theorem

$$\lim_{x \to a} f(x) g(x) = 0.$$