Section 2.4: The Precise definition of a Limit

$\varepsilon - \delta$

DEFINITION 1. *Let $f(x)$ be a function defined for all x in some open interval containing the number a, except possibly at a itself. Then we say that* **the limit of $f(x)$ as x approaches a is L**, *and we write*

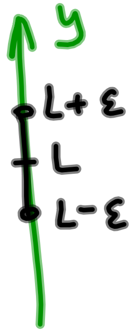
$$\lim_{x \to a} f(x) = L,$$

if for every number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad \underbrace{0 < |x - a| < \delta.}$$

$$-\varepsilon < f(x) - L < \varepsilon$$

$$L - \varepsilon < f(x) < L + \varepsilon$$



$$|x - a| < \delta$$

$$-\delta \overset{+a}{<} x - a \overset{+a}{<} \delta^{+a}$$

$$a - \delta < x < a + \delta$$

$\varepsilon = 0.01$

$\delta = 0.005$

REMARK 2. For a limit from the right we need only assume that $f(x)$ is defined on an interval $(a, b$ extending to the right of a and that the ϵ condition is met for x in an interval $a < x < a + \delta$ extendin to the right of a. A similar adjustment must be made for a limit from the left.

A general form of a limit proof

Assume that we are given a positive number ϵ, and we try to prove that we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

There are two things to do:

1. Preliminary analysis of the problem (guessing a value for δ);

2. Proof (showing that the δ works).

Note that *the value of δ is not unique.* Namely, once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number $\delta_1, \delta_1 < \delta$, will also fulfill those requirements.

EXAMPLE 3. *Use the "epsilon-delta" definition to prove that* $\lim_{x \to 4} \overbrace{(3x-1)}^{f} = 11.$

a L

Show that

for every $\varepsilon > 0$ there exists $\delta > 0$

such that

$|f(x) - L| < \varepsilon$ if $|x - a| < \delta$

$|3x - 1 - 11| < \varepsilon$ if $|x-4| < \delta$

$|3x - 12| < \varepsilon$ if $|x-4| < \delta$

$3|x-4| < \varepsilon$

$|x-4| < \frac{\varepsilon}{3}$ Choose $\delta = \frac{\varepsilon}{3}$

EXAMPLE 4. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 5} x^2 = 25$.

Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$|x^2 - 25| < \varepsilon$ whenever $|x-5| < \delta$

$$|(x+5)(x-5)| < \varepsilon$$

$$|x+5| \cdot |x-5| < \varepsilon$$

$$11|x-5| < \varepsilon$$

$$|x-5| < \frac{\varepsilon}{11}$$

Assume $\delta < 1$

$$|x-5| < 1$$

$$-1 < x-5 < 1$$

$$\underset{+5}{-1+5} < \underset{+5}{x} < \underset{+5}{1+5}$$

$$4 < x < 6$$

$$9 < x+5 < 11$$

$$|x+5| < 11$$

Choose $\delta = \min\left\{\frac{\varepsilon}{11}, 1\right\}$