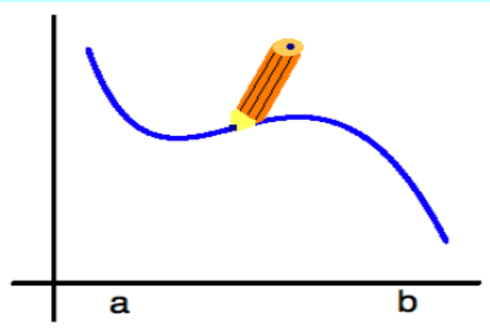
Section 2.5:Continuity

DEFINITION 1. *A function* $f(x)$ *is* **continuous** *at* $x = a$ *if* $\lim_{x\to a} f(x) = f(a)$. *More implicitly: if* f *is continuous at* a *then*

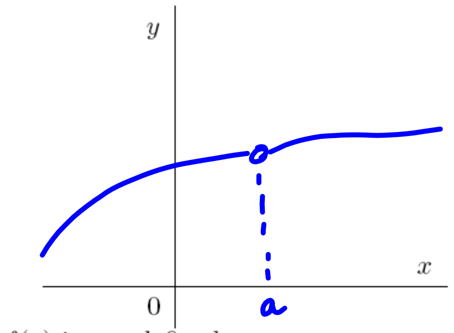
1. $f(a)$ *is defined (i.e.* a *is in the domain of* f*);*
2. $\lim_{x\to a} f(x)$ *exists.* and equals to a finite number
3. $\lim_{x\to a} f(x) = f(a)$.

A function is said to be continuous on the interval $[a, b]$ *if it is continuous at each point in the interval.*

Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

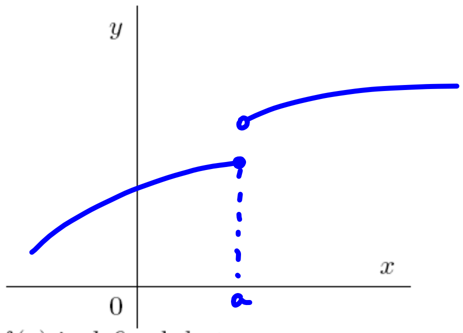


REASONS FOR BEING DISCONTINUOUS:



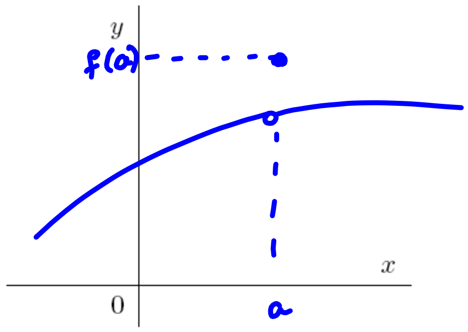
$f(a)$ is not defined

(i.e. a is not in the domain of f)



$f(a)$ is defined, but

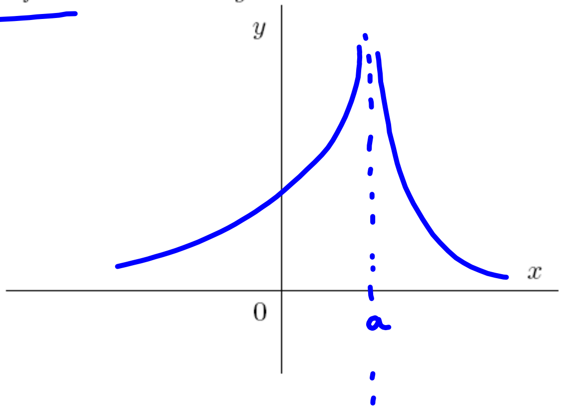
the limit as $x \to a$ DNE



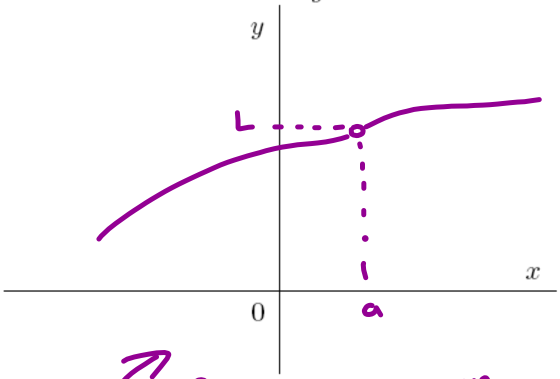
$f(a)$ is defined and $\lim_{x\to a} f(x)$ exists,
but $\lim_{x\to a} f(x) \neq f(a)$

Classification of discontinuities:

infinite discontinuity



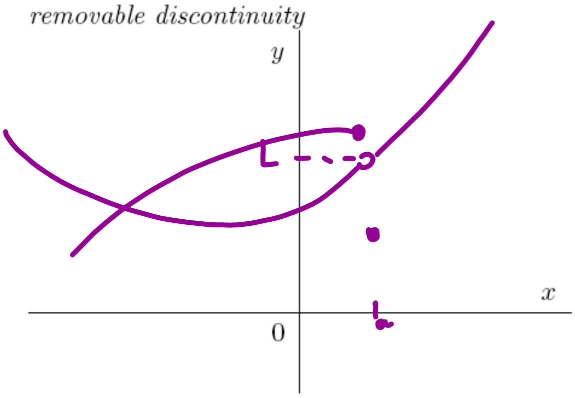
removable discontinuity



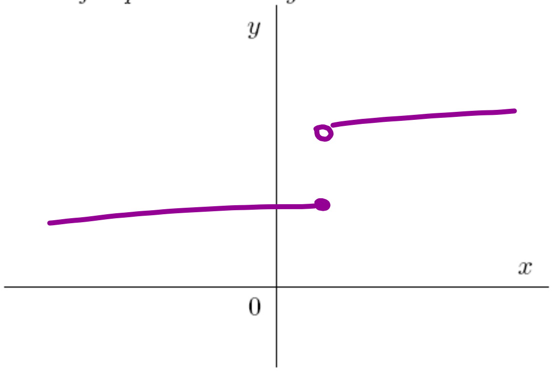
$\lim_{x \to a} f(x) = +\infty$, but $f(a)$ is undefined

One can remove discontinuity at $x = a$ setting $f(a) = L$

removable discontinuity



jump discontinuity



EXAMPLE 2. *Explain why each function is discontinuous at the given point:*

(a) $f(x) = \frac{2x}{x-3}$, $\quad x = 3$ is not in domain of f

In other words, $f(3)$ is undefined

(b) $f(x) = \begin{cases} \frac{x^2-2x+1}{x-1} & \text{if} \quad x \neq 1 \\ 5 & \text{if} \quad x = 1, \end{cases}$ $\quad x = 1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \lim_{x\to 1} \frac{(x-1)^2}{x-1} = \lim_{x\to 1}(x-1) = 0 \neq f(1)=5$$

DEFINITION 3. *A function f is* **continuous from the right at** $x = a$ *if*

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a *if*

$$\lim_{x \to a^-} f(x) = f(a).$$

REMARK 4. *Functions* ***continuous on an interval*** *if it is continuous at every number in the interval.* At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.

EXAMPLE 5. *Find the interval(s) where $f(x) = \sqrt{9 - x^2}$ is continuous.*

Find domain of f : $9 - x^2 \geq 0$

$\sqrt{x^2} \leq \sqrt{9}$

$|x| \leq 3$, or $-3 \leq x \leq 3$

By Direct substitution property for all x in $[-3, 3]$: $\lim_{x \to a} f(x) = f(a)$

EXAMPLE 6. *Find the constant c that makes g continuous on $(-\infty, \infty)$:*

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx^2 - 1 & \text{if } x \geq 4 \end{cases}$$

$y = x^2 - c^2$ and $y = cx^2 - 1$ are continuous for all x.

$$\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x)$$

$\Downarrow$ $x<4$ $\qquad$ $\Downarrow$ $x>4$

$$\lim_{x \to 4^-} (x^2 - c^2) = 16 - c^2 \quad = \quad \lim_{x \to 4^+} (cx^2 - 1) = 16c - 1 = g(4) = 16c - 1$$

$$16 - c^2 = 16c - 1 \qquad ax^2 + bx + c = 0$$

$$c^2 + 16c - 17 = 0 \qquad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$c_1 + c_2 = -16$

$c_1 \cdot c_2 = -17$

$$c_{1,2} = \frac{-16 \pm \sqrt{16^2 + 4 \cdot 17}}{2} = \frac{-16 \pm 18}{2}$$

$$c_1 = \frac{-16 - 18}{2} = -17$$

$$c_2 = \frac{-16 + 18}{2} = 1$$

<u>Answer</u> -17 or 1.

EXAMPLE 7. *For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(a) $f(x) = \dfrac{x^2 - 9}{x^4 - 81}$

$x^4 - 81 = 0 \Rightarrow \underbrace{(x^2 - 9)}_{=0}\underbrace{(x^2+9)}_{\neq 0} = 0$

$\Rightarrow x^2 - 9 = 0 \Rightarrow (x-3)(x+3)=0$

$x=3$ or $x=-3$

$x = \pm 3$ are point of discontinuity

$(-\infty,-3) \cup (-3,3) \cup (3,+\infty)$

$$\lim_{x\to\pm 3} \frac{x^2-9}{x^4-81} = \lim_{x\to\pm3} \frac{\cancel{x^2-9}}{(\cancel{x^2-9})(x^2+9)} = \frac{1}{(\pm 3)^2+9} = \frac{1}{18} \Rightarrow x = \pm 3$$

are removable discontinuities

To remove those discontinuities we define a new function g that agrees with f except of $x = \pm 3$.

$$g(x) = \begin{cases} \dfrac{x^2-9}{x^4-81}, & x \neq \pm 3 \\[2mm] \dfrac{1}{18}, & x = \pm 3 \end{cases}$$

Note that $g(x)$ is continuous for all x.

(b) $f(x) = \dfrac{7}{x+12}$

$x + 12 = 0 \Rightarrow x = -12$

$(-\infty,-12) \cup (-12,\infty)$

$$\lim_{x\to -12} f(x) = \lim_{x\to-12} \frac{7}{x+12}$$ DNE as a finite limit because

$$\lim_{x\to -12^-} \frac{7}{x+12} = -\infty \neq \lim_{x\to-12^+} \frac{7}{x+12} = +\infty$$

Thus $x = -12$ is infinite discontinuity (not removable)

(c) $f(x) = \begin{cases} x^2 + x & \text{if } x < 2 \\ 8 - x & \text{if } x > 2 \\ 4 & \text{if } x = 2 \end{cases}$

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + x) = 2^2 + 2 = 6$

$\Rightarrow \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 8 - x = 8 - 2 = 6$

$\Rightarrow \lim_{x \to 2} f(x) = 6 \neq f(2) = 4$

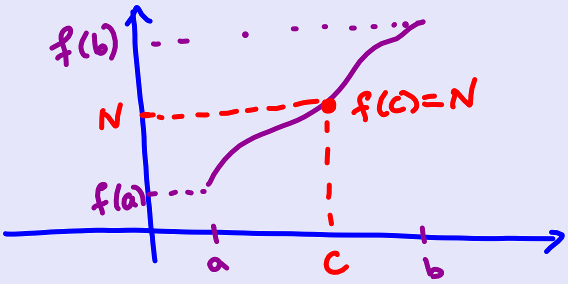
$x = 2$ is removable discontinuity

interval of continuity $(-\infty, 2) \cup (2, +\infty)$

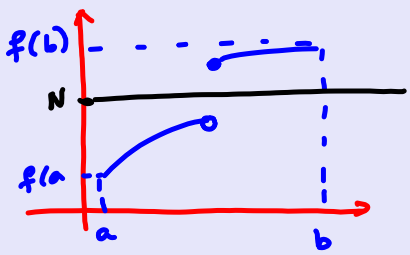
To remove this discontinuity, one can define

$$g(x) = \begin{cases} x^2 + x & , \text{if } x \leq 2 \\ 8 - x & , \text{if } x > 2 \end{cases}$$

Intermediate Value Theorem: If $f(x)$ is continuous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a < c < b$, so that $f(c) = N$.



f is not continuous

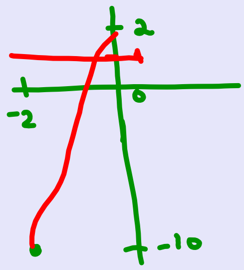


EXAMPLE 8. *If* $f(x) = x^5 - 2x^3 + x^2 + 2$, continuous, *show there* is *a number* c *so that* $f(c) = 1$.

$N = 1$

x	0	1	-1	-2
$f(x)$	2	2	4	$-32+16+4+2=-10$

$f(-2) = -10 \leq 1 \leq 2 = f(0)$



By IVT there exists c such that $-2 < c < 0$ and $f(c) = 1$.

EXAMPLE 9. *Show that following equation has a solution (a root) between* 1 *and* 2*:*

$$3x^3 - 2x^2 - 2x - 5 = 0.$$

To apply IVP we define

$f(x) = 3x^3 - 2x^2 - 2x - 5$ continuous

and show that there exists c such that

$1 < c < 2$ and $f(c) = 0$.

x	1	2	
$f(x)$	<0	>0	

By IVT there exist $1 < c < 2$ such that $f(c) = 0$

(because $f(1) < 0$ and $f(2) > 0$).