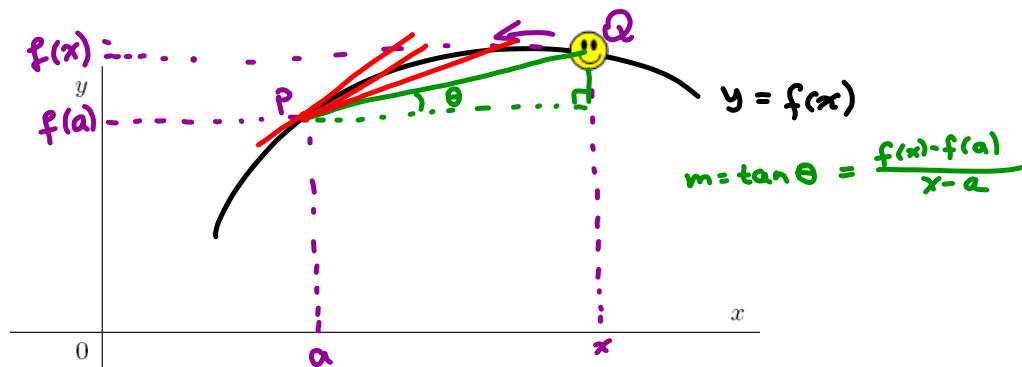


Section 2.7: Tangents, velocities, and other rates of change.

Consider a curve with equation $y = f(x)$ and points $P(a, f(a))$ and $Q(x, f(x))$ on it. The slope of the secant line PQ (also known as average rate or average velocity) is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



$$\lim_{Q \rightarrow P} m_{PQ} = \text{slope of tangent line at } P$$

DEFINITION 1. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{f(a) - f(x)}{a - x} \quad (1)$$

provided that this limit exists.

The slope (1) of tangent line also known as instantaneous rate of change or instantaneous velocity.

EXAMPLE 2. (a) Find the slope of the tangent line to the graph of $f(x) = x^2 - 3x - 4$ at $(5, 6)$. $= a = f(a)$

$$\begin{aligned}
 m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 4 - 6}{x - 5} = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} \\
 &= \lim_{x \rightarrow 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = 5 + 2 = 7
 \end{aligned}$$

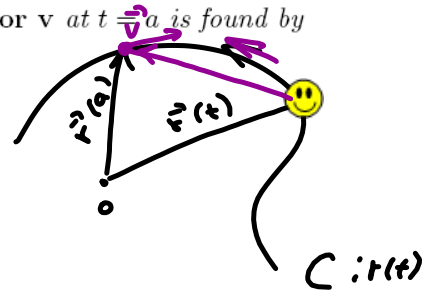
(b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 5$. (Recall that point-slope form for a line through the point (x_1, y_1) with slope m is: $y - y_1 = m(x - x_1)$)

" $(5, 6)$ $m = 7$

$$\boxed{y - 6 = 7(x - 5)}$$

DEFINITION 3. If $\mathbf{r}(t)$ is a vector function then the tangent vector \mathbf{v} at $t = a$ is found by

$$\mathbf{v} = \lim_{t \rightarrow a} \frac{1}{t-a} [\mathbf{r}(t) - \mathbf{r}(a)].$$



EXAMPLE 4. Given curve $\mathbf{r}(t) = \langle 2t, 10t - t^2 \rangle$.

(a) Find a vector tangent to the curve at the point $(4, 16)$.

To find a we need

$$\begin{aligned} \vec{r}(t) &= \langle 4, 16 \rangle \\ \begin{cases} 2t = 4 & \Rightarrow t = 2 \\ 10t - t^2 = 16 & \leftarrow \text{satisfies} \end{cases} \end{aligned}$$

$a=2$ $\vec{r}(2) = \langle 4, 16 \rangle$

$$\begin{aligned} \vec{v} &= \lim_{t \rightarrow 2} \frac{1}{t-2} [\vec{r}(t) - \vec{r}(2)] = \lim_{t \rightarrow 2} \frac{1}{t-2} \langle 2t-4, 10t-t^2-16 \rangle \\ &= \lim_{t \rightarrow 2} \left\langle \frac{2(t-2)}{t-2}, \frac{-(t-2)(t-8)}{t-2} \right\rangle = \langle 2, 6 \rangle \end{aligned}$$

(b) Find parametric equations of the tangent line to $\mathbf{r}(t)$ at $t = 2$.

$$\begin{cases} x = 4 + 2t \\ y = 16 + 6t \end{cases}$$

(c) Find a Cartesian equation of this tangent line.

* Eliminate t

$$\begin{aligned} x &= 4 + 2t \Rightarrow t = \frac{x-4}{2} \\ y &= 16 + 6t \Rightarrow y = 16 + \frac{6(x-4)}{2} \\ &= 16 + 3(x-4) \end{aligned}$$

** Use $y - y_1 = m(x - x_1)$

$$m = \frac{6}{2} = 3 \quad \text{because } \vec{v} = \langle 2, 6 \rangle$$

$$(x_1, y_1) = (4, 16)$$

$$\boxed{y - 16 = 3(x - 4)}$$

Velocities. Denote by $f(t)$ the position of an object at time t .

The **Average Velocity** of the object from $t = a$ to $t = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$


The **Instantaneous Velocity** of the object at time $t = a$ is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

(Compare this formula with (1) by substitution $h = x - a$.)

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \stackrel{h = x - a}{=} \lim_{x \rightarrow a} \frac{f(\cancel{a} + x - \cancel{a}) - f(a)}{x - a}$$

$x = h + a \xrightarrow{h \rightarrow 0} a$



EXAMPLE 5. The position (in meters) of an object moving in a straight path is given by

$$s(t) = t^2 - 2t + 8,$$

where t is measured in seconds.

(a) Find the average velocity over the time interval $[4, 5]$.

$$V_{\text{ave}} = \frac{s(5) - s(4)}{5 - 4} = \frac{25 - 10 + 8 - (16 - 8 + 8)}{1} = 7 \text{ m/s}$$

(b) Find the instantaneous velocity at time $t = 4$.

$$\begin{aligned} v &= \lim_{t \rightarrow 4} \frac{s(t) - s(4)}{t - 4} = \lim_{t \rightarrow 4} \frac{t^2 - 2t + 8 - 16}{t - 4} = \lim_{t \rightarrow 4} \frac{t^2 - 2t - 8}{t - 4} \\ &= \lim_{t \rightarrow 4} \frac{(t - 4)(t + 2)}{t - 4} = 4 + 2 = 6 \text{ m/s} \end{aligned}$$

Other Rates of Change:

The Average Rate of change of function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The Instantaneous Rate of Change of $f(x)$ at $x = a$ is

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

