Section 3.1: Derivative

DEFINITION 1. *The* **Derivative** *of a function* $f(x)$ *at* $x = a$ *is*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Other common notations for the derivative of $y = f(x)$ *are* f', $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$. D f(x)

It follows from the definition that the derivative $f'(a)$ measures:

- The slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$;
- The instantaneous rate of change of $f(x)$ at $x = a$;
- The instantaneous velocity of the object at time at $t = a$ (if $f(t)$ is the position of an object at time t).

EXAMPLE 2. *Given* $f(x) = \dfrac{3}{x+5}$.

(a) *Find the derivative of* $f(x)$ *at* $x = -3$.

$a = -3$

$$f'(-3) = \frac{d}{dx} f(-3) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to -3} \frac{\frac{3}{x+5} - \frac{3}{-3+5}}{x - (-3)}$$

$$= \lim_{x \to -3} \frac{\frac{3}{x+5} - \frac{3}{2}}{x + 3} = \lim_{x \to -3} \frac{3 \cdot 2 - 3(x+5)}{2(x+5)(x+3)} = \lim_{x \to -3} \frac{6 - \overbrace{3x - 15}^{-3x-9}}{2(x+5)(x+3)}$$

$$= \lim_{x \to -3} \frac{-3\cancel{(x+3)}}{2(x+5)\cancel{(x+3)}} = -\frac{3}{2(-3+5)} = -\frac{3}{4}$$

(b) *Find the equation of the tangent line of* $y = f(x)$ *at* $x = -3$.

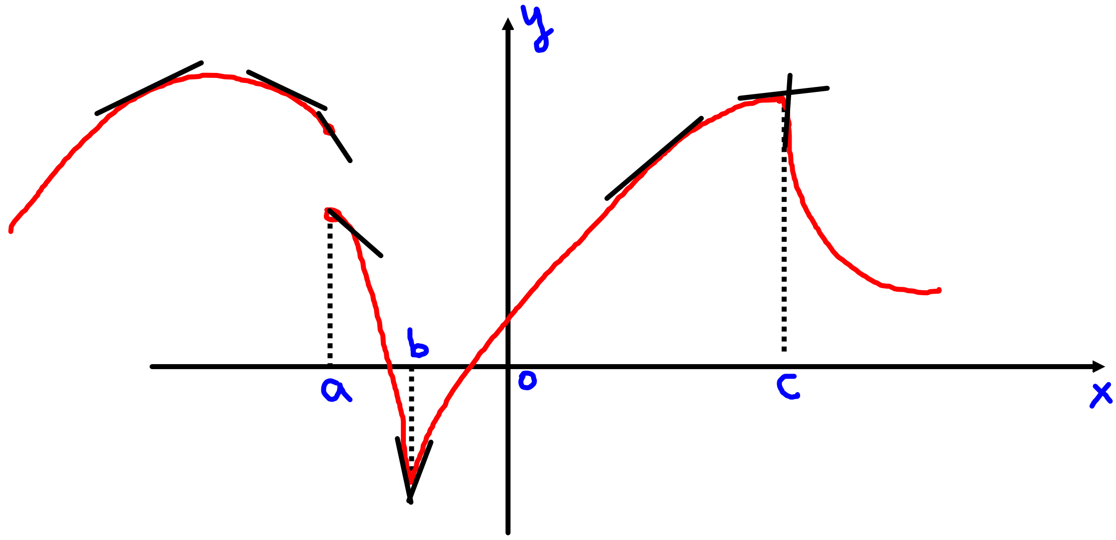
The equation of the tangent line of $y = f(x)$ at $x = a$ is

$$y - f(a) = f'(a)(x - a)$$

In our case $a = -3$, $f(-3) = \frac{3}{2}$, $f'(-3) = -\frac{3}{4}$

$$\boxed{y - \frac{3}{2} = -\frac{3}{4}(x+3)}$$

Question: Where does a derivative not exist for a function?



Answer: $x = a$ (jump discontinuity)
$x = b$ (sharp turn)
$x = c$ (right hand tangent is vertical)

DEFINITION 3. *A function $f(x)$ is said to be* **differentiable** *at $x = a$ if $f'(a)$ exists.*

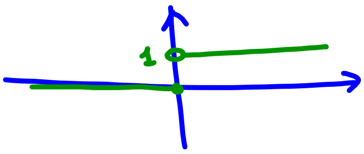
EXAMPLE 4. *Refer to the graph above to determine where f(x) is not differentiable.*

$x = a, b, c$

CONCLUSION: A function $f(x)$ is NOT differentiable at $x = a$ if

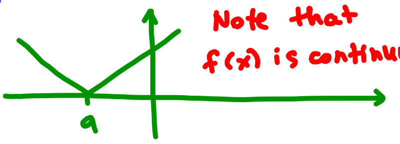
- $f(x)$ is not continuous at $x = a$; $\Rightarrow$ $f'(0)$ DNE

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$$



- $f(x)$ has a sharp turn at $x = a$ (left and right derivatives are not the same);

$$f(x) = |x - a|$$



Note that $f(x)$ is continuous

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

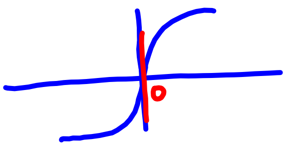
$$= \lim_{h\to 0} \frac{|\cancel{a} + h - \cancel{a}| - \underbrace{|a-a|}_{0}}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} \quad \text{DNE}$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1 \neq \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

- $f(x)$ has a vertical tangent at $x = a$.

$$f(x) = \sqrt[3]{x}$$



$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} \quad \text{DNE}$$

THEOREM 5. *If f is differentiable at $\boxed{a}$ then f is continuous at a.*

$\boxed{\text{Differentiability}} \Longrightarrow \boxed{\text{Continuity}}$

$\not\Longleftarrow$

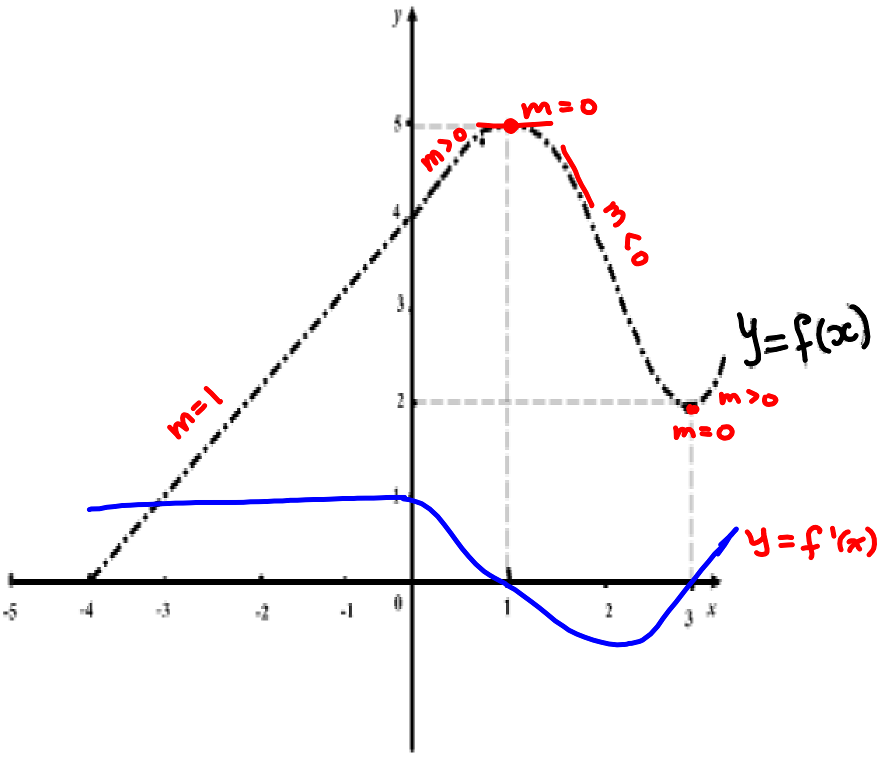
The derivative as a function: If we replace a by x in Definition 1 we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$\left(\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a) \right)$$

A new function $g(x) = f'(x)$ is called the **derivative of** f.

EXAMPLE 6. *Use the graph of $f(x)$ below to sketch the graph of the derivative $f'(x)$.*



m is slope

EXAMPLE 7. *Use the definition of the derivative to find* $f'(x)$ *for* $f(x) = \sqrt{1+3x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{1+3(x+h)} - \sqrt{1+3x}}{h} = \lim_{h \to 0} \frac{\sqrt{1+3x+3h} - \sqrt{1+3x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+3x+3h} - \sqrt{1+3x}}{h} \cdot \frac{(\sqrt{1+3x+3h} + \sqrt{1+3x})}{\sqrt{1+3x+3h} + \sqrt{1+3x}}$$

$$= \lim_{h \to 0} \frac{1+\cancel{3x}+3\cancel{h} - (\cancel{1+3x})}{\cancel{h}\,(\sqrt{1+3x+3h} + \sqrt{1+3x})} = \frac{3}{\sqrt{1+3x+3\cdot 0} + \sqrt{1+3x}} = \frac{3}{2\sqrt{1+3x}}$$

$$(\sqrt{1+3x})' = \boxed{\frac{d(\sqrt{1+3x})}{dx} = \frac{3}{2\sqrt{1+3x}}}$$

EXAMPLE 8. *Each limit below represents the derivative of function $f(x)$ at $x = a$. State f and a in each case.*

(a) $\lim_{h\to 0}\frac{(3+h)^4-81}{h}$ $= \lim_{h\to 0}\frac{(3+h)^4-3^4}{h} = f'(3)$

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ $\boxed{a=3, \quad f=x^4}$

(b) $\lim_{x\to\frac{3\pi}{2}}\frac{\sin x+1}{x-\frac{3\pi}{2}}$ $= \lim_{x\to\frac{3\pi}{2}}\frac{\sin x-(-1)}{x-3\pi/2} = \lim_{x\to\frac{3\pi}{2}}\frac{\sin x-\sin\frac{3\pi}{2}}{x-3\pi/2} = f'(\frac{3\pi}{2})$

$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ $\boxed{f(x)=\sin x, \quad a=3\pi/2}$

$f(a)=f(\frac{3\pi}{2})=\sin\frac{3\pi}{2}=-1$