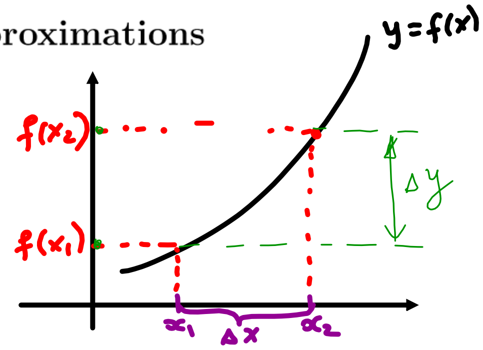
3.11: Differentials; Linear ~~and Quadratic~~ Approximations



Differentials: If x changes from x_1 to x_2, then the *change in* x is

$$\Delta x = x_2 - x_1.$$

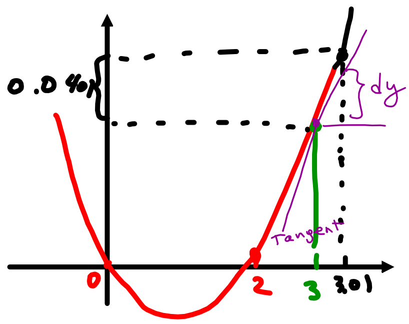
If $y = f(x)$ then corresponding *change in* y is

$$\Delta y = f(x_2) - f(x_1).$$

DEFINITION 1. *Let $y = f(x)$, where f is a differentiable function. Then the differential $\mathrm{d}x$ is an independent variable (i.e. $\mathrm{d}x$ can be given the value of any real number). The differential $\mathrm{d}y$ is then defined in terms of $\mathrm{d}x$ by the equation*

$dx = \Delta x$

$$\boxed{\mathrm{d}y(x) = f'(x)\mathrm{d}x.}$$



EXAMPLE 2. *Compare the values of Δy and $\mathrm{d}y$ if*

$$y = f(x) = x^2 - 2x$$

and x changes from $\underbrace{3}_{x_1}$ to $\underbrace{3.01}_{x_2}$. Illustrate these quantities graphically.

$\Delta x = x_2 - x_1 = 3.01 - 3 = 0.01$

$\Delta y(3) = y(3.01) - y(3) = 3.01^2 - 2\cdot 3.01 - (3^2 - 2\cdot 3) = \underline{0.0401}$

$dy(3) = y'(3)\,dx = y'(3)\,\Delta x = 4\cdot 0.01 = \underline{0.04}$

$y' = 2x - 2 \Rightarrow y'(3) = 2\cdot 3 - 2 = 4$

REMARK 3. Notice that $\mathrm{d}y$ was easier to compute than Δy. For more complicated functions (for example $y = \cos x$) it may be impossible to compute Δy exactly.

CONCLUSION: $\Delta y \approx \mathrm{d}y$ provided we keep Δx small. This yields the following "approximation by differentials" formula.

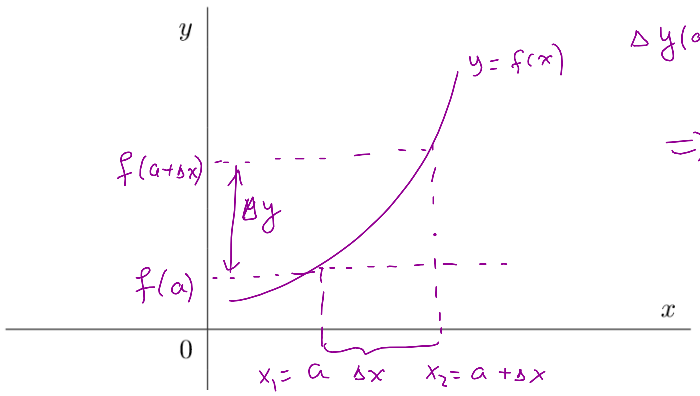
$$f(a + \Delta x) \approx f(a) + \mathrm{d}y = f(a) + \underbrace{f'(a)\Delta x}_{df(a)}.$$

Indeed, assume x changes from $x = a$ to $x = a + \Delta x$. Then

$$\left.\begin{aligned} \Delta y(a) &= f(a+\Delta x) - f(a) \\ \Delta y(a) &\approx df(a) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow f(a+\Delta x) - f(a) \approx df(a)$

$f(a+\Delta x) \approx f(a) + df(a)$



EXAMPLE 4. *Use differentials to find an approximate value for* $\sin 61°$.

$$f(a+\Delta x) \approx f(a) + f'(a)\underbrace{dx}_{\Delta x}$$

$$f(x) = \sin x \implies f'(x) = \cos x$$

$$\sin(a+\Delta x) \approx \sin(a) + \cos(a)\,\Delta x$$

$$\sin(61°) = \sin(60° + 1°) = \sin\Big(\underbrace{\frac{\pi}{3}}_{a} + \underbrace{\frac{\pi}{180}}_{\Delta x}\Big) \approx \sin\frac{\pi}{3} + \Big(\cos\frac{\pi}{3}\Big)\cdot\frac{\pi}{180}$$

$\pi \text{ rad} = 180°$

$\frac{\pi}{180} \text{ rad} = 1°$

$$\approx \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\pi}{180} = \frac{\sqrt{3}}{2} + \frac{\pi}{360} \approx 0.8747$$

using calc. $\sin 61° \approx 0.8746$

EXAMPLE 5. *Use differentials to find an approximate value for* $\sqrt[4]{0.98}$.

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

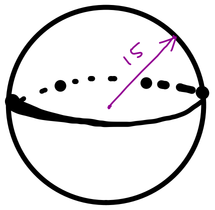
$$f(x) = \sqrt[4]{x} \quad \Rightarrow \quad f'(x) = \left(x^{\frac{1}{4}}\right)' = \frac{1}{4}x^{-3/4}$$

$$\sqrt[4]{0.98} = \sqrt[4]{\underbrace{1}_{a}\underbrace{-0.02}_{\Delta x}} \approx \sqrt[4]{1} + \frac{1}{4}1^{-\frac{3}{4}}(-0.02)$$

$$= 1 - \frac{0.02}{4} = 1 - 0.005 = 0.995$$

cf. Using calculator $\sqrt[4]{0.98} \approx 0.99496$

EXAMPLE 6. *A sphere was measured and its radius was found to be 15 inches with a possible error of no more that 0.02 inches. What is the maximum possible error and what is the relative error in the volume if we use this value of the radius?*



$r = 15 \pm \underbrace{0.02}_{\Delta r}$

$14.98 \le r \le 15.02$

$V(r) = \frac{4}{3}\pi r^3 \Rightarrow V' = 4\pi r^2$

$\Delta V \approx dV = V'(r)\,\Delta r$

$= 4\pi r^2\,\Delta r$

maximum possible error $\Delta V(15) = 4\cdot\pi\, 15^2 \cdot 0.02 \approx 56.5487$ in^3

Explanation

$V(15) = \frac{4}{3}\pi\cdot 15^3 = 14137.17$ in^3

Note: actual volume when $r=15$

$V = 14137.17 \pm 56.57$ in^3

Relative error:

$$\frac{\Delta V(15)}{V(15)} = \left.\frac{4\cancel{\pi}\cancel{r^2}\,\Delta r}{\frac{4}{3}\cancel{\pi}\cancel{r^3}}\right|_{r=15} = \left.\frac{3\,\Delta r}{r}\right|_{r=15} = \frac{3\cdot 0.02}{15} = 0.004$$

Conclusion: the relative error in radius: $\Delta r/r = 0.013 = 1.3\%$
produces a relative error in volume 0.0[redacted]4=[redacted].4%.

Linear Approximation: The function

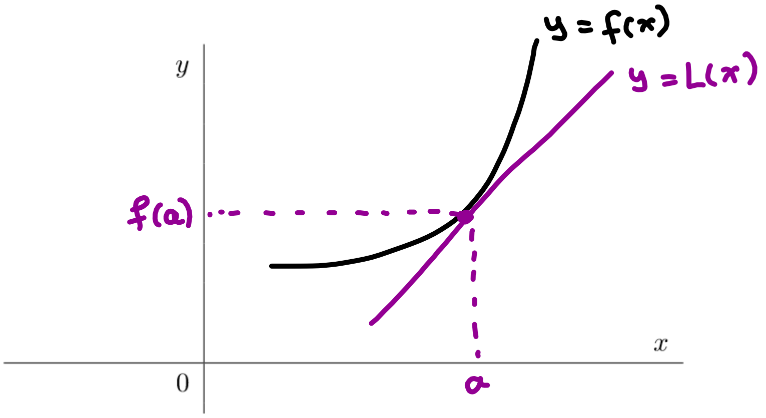
$y=L(x)$

$$\boxed{L_a(x) = f(a) + f'(a)(x-a)}$$

(whose graph is the tangent line to the curve $y = f(x)$ at $(a, f(a))$) is called the **linearization of f at a.** The approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.



x is in a neighborhood of a

$f(x) \simeq L(x)$

EXAMPLE 7. *Determine the linear approximation for* $\sin x$ *at* $a = 0$.

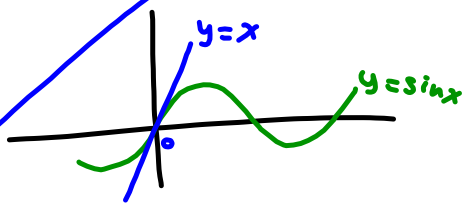
$f(x) = \sin x \quad \Rightarrow \quad f(0) = 0$

$f'(x) = \cos x \quad \Rightarrow \quad f'(0) = 1$

$L(x) = f(a) + f'(a)(x-a)$

$a = 0 \quad \Rightarrow \quad L(x) = f(0) + f'(0)x$

$L(x) = x$



For example

$\sin 0.000007 \approx L(0.000007) = 0.000007$

EXAMPLE 8. *Given* $f(x) = \sqrt[3]{x+1}$.

(a) *Determine linearization for* f *at* $a = 7$.

$f(7) = \sqrt[3]{7+1} = \sqrt[3]{8} = 2$

$f'(x) = \frac{d}{dx}\left((x+1)^{1/3}\right) = \frac{1}{3}(x+1)^{-2/3} \Rightarrow f'(7) = \frac{1}{3} 8^{-2/3} = \frac{1}{3}\left(8^{\frac{1}{3}}\right)^{-2} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$L(x) = f(7) + f'(7)(x-7)$, or $L(x) = 2 + \frac{1}{12}(x-7)$

(b) *Use the linear approximation to approximate the values of* $\sqrt[3]{8.05}$.

$\sqrt[3]{8.05} = f(7.05) \approx L(7.05) = 2 + \frac{1}{12}(7.05 - 7)$

$= 2 + \frac{1}{12} \cdot 0.05 = 2 + \frac{1}{12} \cdot \frac{1}{20} = 2 + \frac{1}{240} = \boxed{\frac{481}{240}}$