Section 3.2: Differentiation formulas

The properties and formulas in this section will be given in both "prime" notation and "fraction" notation.

PROPERTIES:

1. *Constant rule:* If f is a constant function, $f(x) = c$, then $f'(x) = 0$, or $\frac{dc}{dx} = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$$

$$(2013)' = \frac{d}{dx}(2013) = 0$$

2. *Power rule:* If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$, or $\frac{d}{dx}x^n = nx^{n-1}$.

$$(x^{2013})' = 2013\, x^{2013-1} = 2013\, x^{2012}$$

$$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}}$$

3. *Constant multiple rule:* If c is a constant and $f'(x)$ exists then

$$(cf(x))' = cf'(x), \quad \text{or} \quad \frac{d}{dx}(cf) = c\frac{df}{dx}.$$

$$(3\sin x)' = 3(\sin x)'$$

4. *Sum/Difference rule:* If $f'(x)$ and $g'(x)$ exists then

$$(f(x) + g(x))' = f'(x) + g'(x), \quad \text{or} \quad \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}.$$

$$(3e^x - \cos x)' = (3e^x)' - (\cos x)' = 3(e^x)' - (\cos x)'$$

Corollary If c and d are real numbers then

$$(c f(x) + d g(x))' = c f'(x) + d g'(x)$$

5. *Product rule:* If $f'(x)$ and $g'(x)$ exists then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \quad \text{or} \quad \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{dff}{dx}.$$

$$(fg)' = f'g + fg'$$

$$(\underbrace{x}_{f}\underbrace{\sqrt{x}}_{g})' = (x\, x^{\frac{1}{2}})' = (x^{3/2})' = \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}}$$

$$= x'\sqrt{x} + x(\sqrt{x})' = \sqrt{x} + x\frac{1}{2\sqrt{x}}$$

6. *Quotient rule:* If $f'(x)$ and $g'(x)$ exists then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad \text{or} \quad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}.$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

EXAMPLE 1. *Find the derivatives of the following functions:*

(a) $f'(x) = (x^{10} + 3x^5 - 12x + 44 - \pi^5)'$

$$= (x^{10})' + 3(x^5)' - 12x' + (\underbrace{44 - \pi^5}_{\text{constants}})'$$

$$= 10x^{10-1} + 3 \cdot 5x^{5-1} - 12 \cdot 1 \cdot x^{1-1} + 0$$

$$= 10x^9 + 15x^4 - 12$$

(b) $g(t) = (1 + \sqrt{t})^2 = 1 + 2\sqrt{t} + t$ $\quad (a+b)^2 = a^2 + 2ab + b^2$

$$g'(t) = 0 + 2 \cdot \frac{1}{2\sqrt{t}} + 1 = \frac{1}{\sqrt{t}} + 1$$

(c) $F(s) = \left(\frac{s}{3}\right)^4 - s^{-5} = \frac{s^4}{3^4} - s^{-5} = \frac{1}{81}s^4 - s^{-5}$

$$F'(s) = \frac{1}{81}(s^4)' - (s^{-5})' = \frac{4s^3}{81} - (-5)s^{-5-1}$$

$$= \frac{4s^3}{81} + 5s^{-6}$$

(d) $y = \dfrac{u^5+1}{u^2\sqrt{u}}$

use Quotient Rule

or

$$y = \frac{u^5+1}{u^2 u^{1/2}} = \frac{u^5+1}{u^{5/2}}$$

$$y = \frac{u^5}{u^{5/2}} + \frac{1}{u^{5/2}} = u^{\frac{5}{2}} + u^{-\frac{5}{2}}$$

$$y' = \left(u^{\frac{5}{2}}\right)' + \left(u^{-\frac{5}{2}}\right)' = \frac{5}{2}u^{\frac{5}{2}-1} + \left(-\frac{5}{2}\right)u^{-\frac{5}{2}-1}$$

$$y' = \frac{5}{2}u^{\frac{3}{2}} - \frac{5}{2}u^{-\frac{7}{2}}$$

(e) $f(x) = \underbrace{(x^4 - 3x^2 + 11)}_{F(x)}\underbrace{(3x^3 - 5x^2 + 22)}_{G(x)}$

use Product Rule

$$f'(x) = (FG)' = F'G + FG' = (x^4-3x^2+11)'G + F(3x^3-5x^2+22)'$$

$$= (4x^3-6x)(3x^3-5x^2+22) + (x^4-3x^2+11)(9x^2-10x)$$

(f) $g(z) = \dfrac{4-z^2}{4+z^2} = \dfrac{F}{G}$ use Quotient Rule

$$g' = \frac{F'G - FG'}{G^2} = \frac{(4-z^2)'G - F(4+z^2)'}{G^2}$$

$$= \frac{-2z(4+z^2) - (4-z^2)\cdot 2z}{(4+z^2)^2}$$

$$= \frac{-2z\left[4+\cancel{z^2}+4-\cancel{z^2}\right]}{(4+z^2)^2} = -\frac{16z}{(4+z^2)^2}$$

EXAMPLE 2. *The functions f and g satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	5	12
3	1	2	-2	8

Find the indicated quantity:

(a) $h'(3)$ *if* $h(x) = (3x^2+1)g(x)$

Product Rule

$$h'(x) = \big(\underbrace{(3x^2+1)}\underbrace{g(x)}\big)' = \underbrace{(3x^2+1)'}g(x) + (3x^2+1)g'(x)$$

$$= 6x\, g(x) + (3x^2+1)\, g'(x)$$

$$h'(3) = 6\cdot 3\, g(3) + (3\cdot 3^2+1)\, g'(3) = 18\cdot(-2) + 28\cdot 8 = 188$$

(b) $H'(1)$ *if* $H(x) = \dfrac{x^2}{f(x)}$

Quotient Rule

$$H'(x) = \frac{(x^2)' f(x) - x^2 f'(x)}{f^2(x)} = \frac{2x\, f(x) - x^2 f'(x)}{f^2(x)}$$

$$H'(1) = \frac{2f(1) - f'(1)}{f^2(1)} = \frac{2\cdot(-5) - 8}{(-5)^2} = -\frac{18}{25}$$

EXAMPLE 3. *Given* $f(x) = x^3 - 5x^2 + 6x - 3$

(a) *Find the equation of the tangent line to the graph of* $f(x)$ *at the point* $(1, -1)$.

Tangent at $(a, f(a))$:

$$y - f(a) = f'(a)(x-a)$$

In our case $a = 1$, $f(a) = f(1) = -1$

$$f'(x) = 3x^2 - 10x + 6 \Rightarrow f'(1) = 3 - 10 + 6 = -1$$

$$y - (-1) = -1(x-1) \Rightarrow y + 1 = -x + 1$$

$$\boxed{y = -x}$$

(b) *Find the value(s) of* x *where* $f(x)$ *has* $\underbrace{\textit{a tangent line}}_{m_1 = f'(x)}$ *that is parallel to* $\underbrace{y = 6x + 1}_{m_2 = 6}$.

Two lines are parallel if and only if $m_1 = m_2$

$$f'(x) = 6$$

$$3x^2 - 10x + 6 = 6$$

$$3x^2 - 10x = 0$$

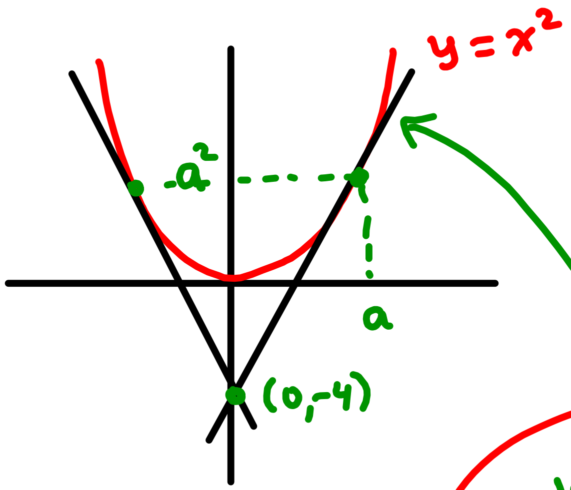
$$x(3x - 10) = 0$$

$x = 0$ OR $3x - 10 = 0$

$$x = \frac{10}{3}$$

Answer: $x = 0, \frac{10}{3}$

EXAMPLE 4. *Show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$ and find their equations.*



(a, a^2) tangent point

Tangent to $y = x^2$ at $\underbrace{(a, a^2)}$

has slope $m = y'(a) = \underbrace{2a}$

Tangent line equation

$$y - a^2 = 2a(x - a)$$

We know that $(0,-4)$ belongs to this line:

$$-4 - a^2 = 2a(0 - a)$$

$$-4 - a^2 = -2a^2$$

$$-4 = -a^2 \Rightarrow a^2 = 4 \Rightarrow \boxed{a = \pm 2}$$

$a = 2$

$a = -2$

$$\boxed{\begin{array}{l} y - 4 = 4(x - 2) \\ y - 4 = -4(x + 2) \end{array}}$$

EXAMPLE 5. *Let* $f(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$ $\Rightarrow f'(x) = \begin{cases} -2, & x<-1 \\ 2x, & -1<x<1 \\ 1, & x>1 \end{cases}$

(a) *Give a formula for* f'.

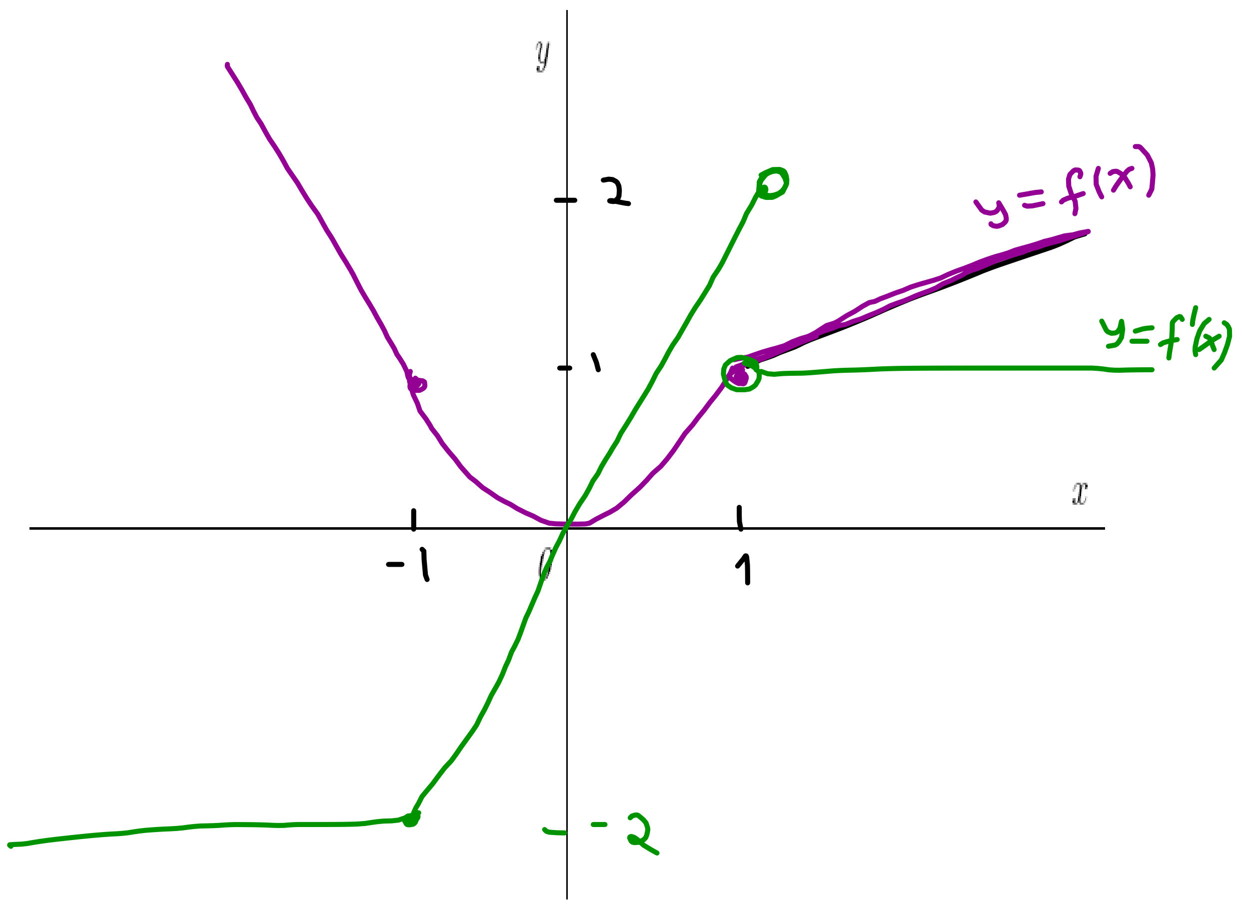
$$\left.\begin{array}{l} f'_-(-1) = -2 \\ f'_+(-1) = 2\cdot(-1) = -2 \end{array}\right\} \Rightarrow f'(-1) = -2$$

$$f'(x) = \begin{cases} -2, & x \le -1 \\ 2x, & -1<x<1 \\ 1, & x>1 \end{cases}$$

$$\left.\begin{array}{l} f'_-(1) = 2\cdot 1 = 2 \\ f'_+(1) = 1 \end{array}\right\} \Rightarrow f'(1) \text{ DNE}$$

(b) *For what value(s) of* x *the function is not differentiable?* $x = 1$

(c) *Sketch the graph of f and f' on the same axis.*



EXAMPLE 6. *A ball is thrown into the air. Its position at time t is given by*

$$\vec{r}(t) = \langle 2t, 10t - t^2 \rangle .$$

(a) *Find the velocity of the ball at time $t = 2$.*

$\vec{v}(t) = \vec{r}\,'(t) = <2, 10-2t>$

$\vec{v}(2) = \vec{r}\,'(2) = <2, 10-4> = <2,6>$

(b) *Find the speed of the ball at time $t = 2$.*

$\text{speed}\Big|_{t=2} = |\vec{v}(2)| = |<2,6>| = \sqrt{2^2+6^2} = \sqrt{40}$

EXAMPLE 7. *Let*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 2 \\ ax + b & \text{if} \quad x > 2 \end{cases}$$

Find the values of a and b that makes f differentiable everywhere.

It follows immediately that f is differentiable for all $x \neq 2$. It remains to consider $x=2$:

We need $f'_-(2) = f'_+(2) \Rightarrow a = 4$

$f'_-(2) = 2x\big|_{x=2} = 4$, $\quad f'_+(2) = a$

In addition we need continuity. Namely,

$f_-(2) = f_+(2)$

$f_-(2) = x^2\big|_{x=2} = 4$, $\quad f_+(2) = 4x+b\big|_{x=2} = 8+b$

$4 = 8 + b \Rightarrow b = -4$

Answer: $a = 4$, $b = -4$