3.4 Derivatives of Trigonometric Functions

It is important to remember that everything for six trigonometric functions ($\sin x, \cos x, \tan x, \cot x, \csc x, \sec x$ will be done in radians.

EXAMPLE 1. *Compute:*

(a) $\lim_{x\to 0} \sin x =$ $\sin 0 = 0$ (b) $\lim_{x\to 0} \cos x =$ $\cos 0 = 1$

THEOREM 2.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0.}$$

Proof

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} -\frac{\sin^2 x}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1} = -\underbrace{\lim_{x\to 0} \frac{\sin x}{x}} \cdot \lim_{x\to 0} \frac{\sin x}{\cos x + 1}$$

$$= -1 \cdot \frac{0}{1+1} = 0$$

EXAMPLE 3. *Find these limits:*

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x} = \lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot 5 = 5\lim_{x\to 0}\frac{\sin(5x)}{5x}$

$u = 5x \underset{x\to 0}{\longrightarrow} 0$ $\displaystyle = 5\lim_{u\to 0}\frac{\sin u}{u} = 5\cdot 1 = \boxed{5}$

(c) $\displaystyle\lim_{x\to 0}\frac{x}{\sin(4x)} = \lim_{x\to 0}\frac{1}{\frac{\sin 4x}{x}} = \frac{1}{\lim_{x\to 0}\frac{\sin 4x}{x}} = \frac{1}{4}$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(9x)}{\sin(7x)} = \lim_{x\to 0}\frac{\sin(9x)}{x}\cdot\frac{x}{\sin(7x)} = \lim_{x\to 0}\frac{\sin 9x}{x}\cdot\lim_{x\to 0}\frac{x}{\sin 7x} = 9\cdot\frac{1}{7} = \frac{9}{7}$

Conclusion: If $a, b \neq 0$ then

$$\lim_{x\to 0}\frac{\sin(ax)}{x} = a, \qquad \lim_{x\to 0}\frac{x}{\sin(ax)} = \frac{1}{a}, \qquad \lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$$

(d) $\lim_{x\to 0} \frac{1}{x^2 \cot^2(3x)} = \lim_{x\to 0} \frac{1}{x^2 \frac{\cos^2 3x}{\sin^2 3x}} = \lim_{x\to 0} \frac{\sin^2(3x)}{x^2 \cos^2 3x}$

$$= \underbrace{\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)^2}_{\left(\lim_{x\to 0}\frac{\sin 3x}{x}\right)^2} \cdot \underbrace{\lim_{x\to 0} \frac{1}{\cos^2 3x}}_{\frac{1}{1^2}} = 3^2 \cdot 1 = 9$$

(e) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{x}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \lim_{x\to 0} \frac{x}{\sin x}$

$$= 0 \cdot 1 = 0$$

EXAMPLE 4. *Find the following derivatives:*

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$

$= \lim_{h\to 0} \frac{\sin x \cosh + \sin h \cos x - \sin x}{h}$

$= \lim_{h\to 0} \frac{\overset{\sin x(\cos h-1)}{\sin x \cos h - \sin x}}{h} + \lim_{h\to 0} \frac{\sin h}{h}\cos x = \sin x \lim_{h\to 0}\frac{\cos h-1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$

$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$

Remark Similarly one can get $(\cos x)' = -\sin x$. (Please try to get it)

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \left(\frac{\sin x}{\cos x}\right)' \overset{Q.R.}{=} \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$ $\quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Derivatives of Trig Functions (memorize these!)

$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$	$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$	$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$	$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$

EXAMPLE 5. *Find the derivative of these functions.*

(a) $y = \cot x + 5\sec x + x\sqrt{x}$ $x^{3/2}$

$$y' = -\csc^2 x + 5\sec x \tan x + \frac{3}{2}x^{\frac{1}{2}}$$

(b) $f(x) = \dfrac{\cos x}{1 + \sin x}$ Q.R.

$$f'(x) = \frac{(\cos x)'(1+\sin x) - \cos x(1+\sin x)'}{(1+\sin x)^2}$$

$$= \frac{-\sin x\,(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= -\frac{\sin x + \overbrace{\sin^2 x + \cos^2 x}^{1}}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$$

EXAMPLE 6. *Find the equation of the tangent line to the graph of function* $y = x^2 \sin x$ *at* $x = \frac{\pi}{4}$.

$$y - f(a) = f'(a)(x-a)$$

$$a = \frac{\pi}{4}$$

$$f'(x) = (x^2 \sin x)' = 2x \sin x + x^2 \cos x$$

$$f'\left(\frac{\pi}{4}\right) = 2\frac{\pi}{4} \underbrace{\sin \frac{\pi}{4}} + \frac{\pi^2}{16} \underbrace{\cos \frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{2} + \frac{\pi^2}{16}\right)$$

$$f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot \sin \frac{\pi}{4} = \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2} = \frac{\pi^2\sqrt{2}}{32}$$

$$y - \frac{\pi^2\sqrt{2}}{32} = \frac{\sqrt{2}}{2}\left(\frac{\pi}{2} + \frac{\pi^2}{16}\right)\left(x - \frac{\pi}{4}\right)$$