Section 3.5: Chain Rule

Question: How to find the derivatives of the following functions:

$$y = (\underbrace{x^6 + 4x^2 + 12}_{\text{inner}})^{15}; \qquad y = \sec(12x^2) + \tan^3(x) \qquad y = \sqrt[3]{4+x}$$

If $f(x) = x^{15}$ and $g(x) = x^6 + 4x^2 + 12$ then $f(g(x))$

Review of Composite Functions:

$$[f \circ g](x) = f(\overbrace{g(x)}^{\text{inner function}})$$

outer function

$$[f \circ g](x) = (x^6 + 4x^2 + 12)^{15}$$

Conversely, if $[f \circ g](x) = \sec(12x^2)$ then $f(x) =$ $\sec x$ and $g(x) =$ $12x^2$

The CHAIN RULE: *If the derivatives* $g'(x)$ *and* $f'(x)$ *both exist, and* $F = f \circ g$ *is the composite defined by*

$$F(x) = f(\underbrace{g(x)}_{\text{inner}})$$

f
$\downarrow$
g
$\downarrow$
x

then

$$F'(x) = f'(g(x))g'(x).$$

In Leibniz notation: If the derivatives of $y = f(u)$ *and* $u = g(x)$ *both exist then*

$$y = f(g(x)) = f(u)$$

is differentiable function of x *and*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. = f' \cdot u'$$

$y = \cancel{f(u)}\ f(u(x))$	$u(x)$ inner	$f(u)$ outer	$\frac{dy}{dx} = y' \cdot u'$
$y = \underbrace{(x^6+4x^2+12)}_{u}{}^{15}$	$u = x^6+4x^2+12$ $u' = 6x^5+8x$	$y = u^{15}$ $y' = 15u^{14}$	$\frac{dy}{dx} = 15u^{14}(6x^5+8x)$ $= 15(x^6+4x^2+12)^{14}(6x^5+8x)$
$y = \sec(12x^2)$	$u = 12x^2$ $u' = 24x$	$y = \sec u$ $y' = \sec u \tan u$	$\frac{dy}{dx} = \sec(12x^2)\tan(12x^2)\cdot 24x$
$y = \tan^3(x)$ $= (\tan x)^3$	$u = \tan x$ $u' = \sec^2 x$	$y = u^3$ $y' = 3u^2$	$\frac{dy}{dx} = 3\tan^2 x \sec^2 x$
$y = \sqrt[3]{4+x}$	$u = 4+x$ $u' = 1$	$y = \sqrt[3]{u} = u^{\frac{1}{3}}$ $y' = \frac{1}{3}u^{-2/3}$	$\frac{dy}{dx} = \frac{1}{3}(4+x)^{-2/3}$ $= \frac{1}{3\sqrt[3]{(4+x)^2}}$
$y = [g(x)]^n$	$u = g(x)$ $u' = g'(x)$	$y = u^n$ $y' = nu^{n-1}$	$\frac{dy}{dx} = n(g(x))^{n-1}g'(x)$ Generalized Power Rule

EXAMPLE 1. *Find the derivative:*

(a) $f(x) = \dfrac{1}{(x^3+5x^2+12)^{2012}} = (\underbrace{x^3+5x^2+12})^{-2012}$

By Generalized Power Rule $(n=-2012)$

$$f'(x) = -2012\,(x^3+5x^2+12)^{-2012-1}\,(x^3+5x^2+12)'$$

$$f'(x) = -2012\,(x^3+5x^2+12)^{-2013}\,(3x^2+10x)$$

(b) $h(x) = x^8(3\sqrt{x}-11)^8 = \left(x\,(3\sqrt{x}-11)\right)^8 = (3x\sqrt{x}-11x)^8$

$$h(x) = (\underbrace{3x^{\frac{3}{2}}-11x})^8$$

$$h'(x) = 8\,(3x^{3/2}-11x)^7\,(3x^{\frac{3}{2}}-11x)'$$

$$= 8\,(3x^{3/2}-11x)^7\left(\tfrac{9}{2}x^{\frac{1}{2}}-11\right)$$

(c) $f(x) = \cos(5x) + \cos^5 x = \underbrace{\cos}_{\text{outer}}(5x) + (\underbrace{\cos x}_{\text{inner}})^5$

$$f'(x) = -\sin(5x)\cdot(5x)' + 5(\cos x)^4(\cos x)'$$

$$= -5\sin(5x) - 5\cos^4 x\,\sin x$$

(d) $f(x) = \sqrt{\underbrace{x^3+\sqrt{x^2+\sqrt{x}}}_{\text{☺}}}$ $\qquad \left(\sqrt{\text{☺}}\right)' = \dfrac{1}{2\sqrt{\text{☺}}}(\text{☺})'$

$$f'(x) = \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}\left(x^3+\sqrt{x^2+\sqrt{x}}\right)'$$

$$= \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}\left(3x^2 + \left(\sqrt{\underbrace{x^2+\sqrt{x}}_{\text{☺}}}\right)'\right)$$

$$= \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}\left(3x^2 + \frac{1}{2\sqrt{x^2+\sqrt{x}}}\cdot(x^2+\sqrt{x})'\right)$$

$$= \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}\left(3x^2 + \frac{1}{2\sqrt{x^2+\sqrt{x}}}\left(2x + \frac{1}{2\sqrt{x}}\right)\right)$$

EXAMPLE 2. *Find F' and G' if*

$$F(x) = f(\underbrace{\sin x}_{\text{inner}}), \qquad G(x) = \overbrace{\sin}^{\text{outer}}(f(x)),$$

where $f(x)$ is a differentiable function.

$F(x) = f(\underbrace{\sin x}_{u}) = f(u(x))$

$F'(x) = \frac{df}{du} u'(x) = \frac{df}{du} \cos x$

where $u(x) = \sin x$

$G(x) = \sin[f(x)]$

$G'(x) = \cos(f(x))\, f'(x)$

Illustration

$F(x) = \tan(\underbrace{\sin x}_{u})$

$F'(x) = \sec^2 u \; \cos x = \sec^2(\sin x)\cos x$

Illustration :

$G(x) = \sin(\overbrace{\tan x}^{f(x)})$

$G'(x) = \cos(\tan x)\,(\tan x)'$

$= \cos(\tan x) \cdot \sec^2 x$

EXAMPLE 3. *Let $f(x)$ and $g(x)$ be given differentiable functions satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	3	12
3	1	2	-2	8

Suppose that $h = f \circ g$. Find $h'(1)$.

$h(x) = f(g(x))$. By Chain Rule

$h'(x) = f'(g(x))\, g'(x)$

$h'(1) = f'(g(1))\, g'(1) = f'(3) \cdot 12 = 2 \cdot 12 = \boxed{24}$