3.6: Implicit differentiation

EXAMPLE 1. *Find y' if the $y = y(x)$ satisfies the equation $xy = 5$ for all values of x in its domain and evaluate $y'(5)$.*

Solution 1 (by explicit differentiation):

$$xy = 5 \Rightarrow y = \frac{5}{x} \Rightarrow y' = -\frac{5}{x^2} \Rightarrow y'(5) = -\frac{5}{5^2} = \boxed{-\frac{1}{5}}$$

Solution 2 (by implicit differentiation):

$$xy = 5$$

$$\frac{d}{dx}\left(x\ y(x)\right) = \frac{d}{dx}(5)$$

$$y + x y' = 0 \Rightarrow y'(x) = -\frac{y(x)}{x}$$

$$y'(5) = -\frac{y(5)}{5} = \boxed{-\frac{1}{5}}$$

$$xy = 5 \Rightarrow y = \frac{5}{x}$$

$$y(5) = \frac{5}{5} = 1$$

EXAMPLE 2. (a) *If* $x^2 + y^2 = 16$ *find* $\frac{\mathrm{d}y}{\mathrm{d}x}$. $y = y(x)$

$$x^2 + [y(x)]^2 = 16$$

Differentiate both sides of this EQUATION w.r.t. x

(using Chain Rule and other differentiation technique):

$$\frac{d}{dx}\left[x^2 + [y(x)]^2\right] = \frac{d}{dx}[16]$$

$$2x + 2y\,y' = 0 \implies \boxed{y'(x) = -\frac{x}{y}},\ y \neq 0$$

(b) *Find the equation of the tangent line to* $x^2 + y^2 = 16$ *at the point* $(\overset{x}{2}, \overset{y}{2\sqrt{3}})$.

$$y - 2\sqrt{3} = y'(2)\,(x - 2)$$

$$y'(2) = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\boxed{y - 2\sqrt{3} = -\frac{1}{\sqrt{3}}(x-2)}$$

EXAMPLE 3. *Find* $\frac{\mathrm{d}y}{\mathrm{d}x}$ *for the following:*

$y = y(x)$

(a) $4x^3 + 2y^2 = 4xy^5 + y$

$$\frac{d}{dx}\left(4x^3 + 2[y(x)]^2\right) = \frac{d}{dx}\left[4x\,[y(x)]^5 + y(x)\right]$$

Apply P.R. C.R

$$12x^2 + 2\cdot 2yy' = 4y^5 + 4x\cdot 5y^4y' + y'$$

$$4yy' - 20xy^4y' - y' = 4y^5 - 12x^2$$

Factor:

$$y'\left(4y - 20xy^4 - 1\right) = 4y^5 - 12x^2$$

$$y' = \frac{4y^5 - 12x^2}{4y - 20xy^4 - 1}$$

$y = y(x)$

(b) $x^3 - \cot(xy^2) = x\cos y$

$$\frac{d}{dx}\left(x^3 - \cot\left(x\underbrace{\left(y(x)\right)^2}\right)\right) = \frac{d}{dx}\left(x \cos y(x)\right)$$

$$3x^2 - \left(-\csc^2(xy^2)\right)\underbrace{\frac{d}{dx}\left(x\left[y(x)\right]^2\right)}_{P.R.} = \cos y + x(-\sin y)\,y'$$

$$3x^2 + \csc^2(xy^2)\left(y^2 + x\,2yy'\right) = \cos y - x(\sin y)y'$$

$$3x^2 + y^2\csc^2(xy^2) + \underline{2xy\csc^2(xy^2)}\,y' = \cos y - \underline{x\sin y\, y'}$$

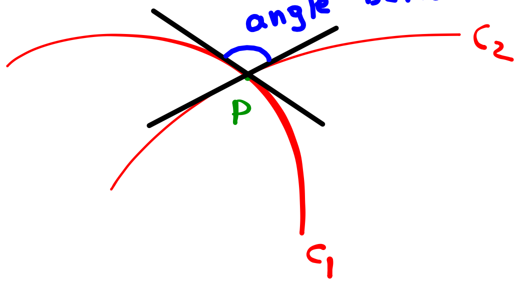
$$2xy\csc^2(xy^2)\,\underline{y'} + x\sin y\,\underline{y'} = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$y'\left(2xy\csc^2(xy^2) + x\sin y\right) = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$y' = \frac{\cos y - 3x^2 - y^2\csc^2(xy^2)}{2xy\csc^2(xy^2) + x\sin y}$$

DEFINITION 4. *Two curves are said to be* **orthogonal** *if at the point(s) of their intersection, their tangent lines are orthogonal(perpendicular). In this case we also say that the angle between these curves is* $\frac{\pi}{2}$.

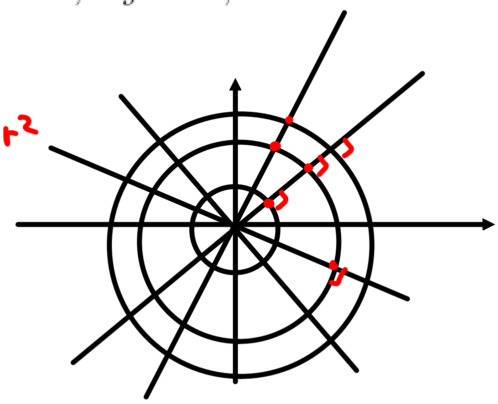
angle between C_1 and C_2 at P.



`Illustration:` Consider two families of curves:

$$x^2 + y^2 = r^2, \quad y = kx,$$

where r and k are real parameters.



m_1 slope of tangent line to $x^2+y^2=r^2$

By Ex. 2

$m_1 = y'(x) = -\frac{x}{y}$

m_2 slope of tangent line to $y=kx$

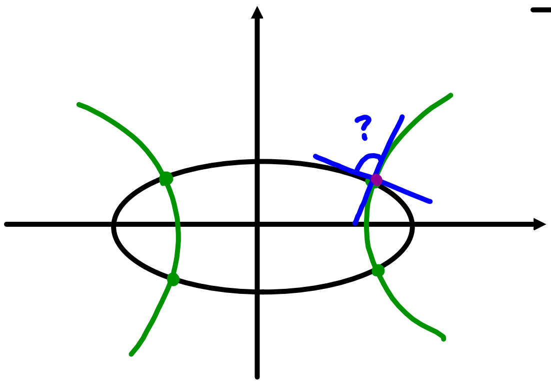
then $m_2 = k = \frac{y}{x}$

$m_1 \cdot m_2 = -\frac{x}{y} \cdot \frac{y}{x} = -1 \Rightarrow$ tangent are perpendicular

$\Rightarrow$ these two families are orthogonal at their intersection points.

EXAMPLE 5. *Are these curves orthogonal?* at intersection points

$$\underbrace{x^2 - y^2 = 5}_{\text{hyperbola}}, \quad \underbrace{4x^2 + 9y^2 = 72}_{\text{ellipse}}$$



Find intersection points:

$$\begin{cases} x^2 - y^2 = 5 \\ 4x^2 + 9y^2 = 72 \end{cases} \Rightarrow x^2 = 5 + y^2$$

$$4(5+y^2) + 9y^2 = 72$$

$$20 + 4y^2 + 9y^2 = 72$$

$$13y^2 = 52$$

$$y^2 = \frac{52}{13} = 4$$

$$y = \pm 2$$

$$x^2 = 5 + 4$$

$$x^2 = 9$$

$$x = \pm 3$$

4 intersection points $(\pm 3, \pm 2)$

Find slopes of tangent lines to given curves:

$x^2 - y^2 = 5$ | $4x^2 + 9y^2 = 72$

Use implicit differentiation:

$\frac{d}{dx}\left(x^2 - (y(x))^2\right) = \frac{d}{dx}5$ | $\frac{d}{dx}\left(4x^2 + 9(y(x))^2\right) = \frac{d}{dx}72$

$2x - 2yy' = 0$ | $8x + 18yy' = 0$

$m_1 = y' = \frac{x}{y}$ | $m_2 = y' = -\frac{8x}{18y} = -\frac{4x}{9y}$

We know that two lines are perpendicular if and only if $m_1 \cdot m_2 = -1$.

In our case,

$$m_1 \cdot m_2 = \frac{x}{y} \cdot \left(-\frac{4x}{9y}\right) = -\frac{4x^2}{9y^2} \underset{\text{at } (\pm 3, \pm 2)}{=}$$

$$= -\frac{4 \cdot 9}{9 \cdot 4} = -1$$

Hence, the given curves are orthogonal.