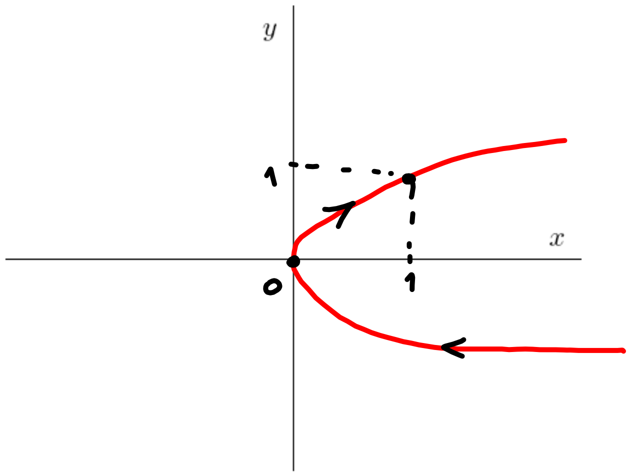
3.7: Derivatives of the vector functions

direction of motion

EXAMPLE 1. *Sketch the curve* $\mathbf{r}(\mathbf{t})$ *and indicate with arrow the direction in which t increases if*

(a) $\mathbf{r}(\mathbf{t}) = \langle t^2, t \rangle$ Eliminate parameter :

$$\begin{cases} x = t^2 \\ y = t \end{cases} \Rightarrow \boxed{x = y^2}$$



$\vec{r}(0) = \langle 0, 0 \rangle$

$\vec{r}(1) = \langle 1, 1 \rangle$

(b) $\mathbf{r}(\mathrm{t}) = \langle 2\sin t, \cos t\rangle$

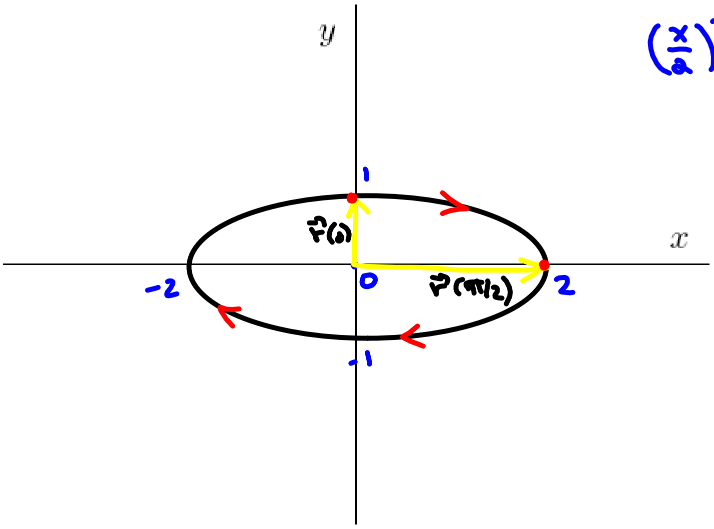
$x = 2\sin t \Rightarrow \sin t = \frac{x}{2}$

$y = \cos t$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{2}\right)^2 + y^2 = 1$

$\frac{x^2}{4} + y^2 = 1$ ellipse



$\vec{r}(0) = \langle 0, 1\rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 2, 0)$

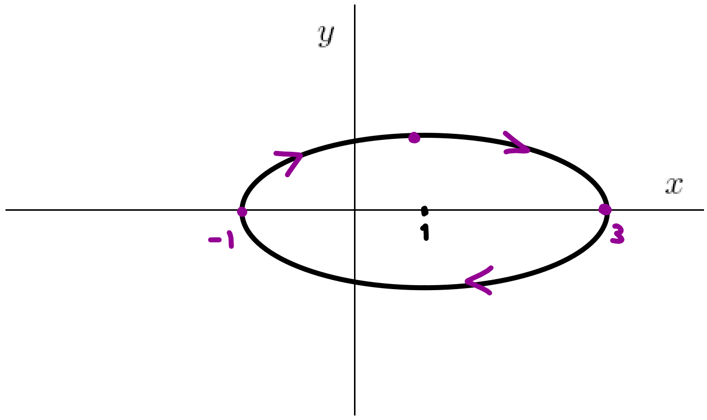
(c) $\mathbf{r}(\mathrm{t}) = \langle 1 + 2\sin t, \cos t \rangle$

$$\begin{cases} x = 1 + 2\sin t \\ y = \cos t \end{cases} \Rightarrow \frac{x-1}{2} = \sin t$$

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x-1}{2}\right)^2 + y^2 = 1$$

$$\frac{(x-1)^2}{4} + y^2 = 1$$



$\vec{r}(0) = \langle 1, 1 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 3, 0 \rangle$

DEFINITION 2. *If* $\vec{r}(t) = \langle x(t), y(t) \rangle$ *is a vector function, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

if both $x'(t), y'(t)$ *exist.*

EXAMPLE 3. *If* $\mathbf{r}(\mathbf{t}) = \langle t^2, \sqrt{t-5} \rangle$ *find the domain of* $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$.

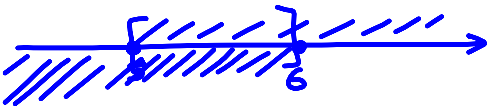
$$\left.\begin{aligned} &D(t^2) = \mathbb{R} \\ &D(\sqrt{t-5}) = [5, +\infty) \end{aligned}\right\} \Rightarrow D(\vec{r}(t)) = [5, +\infty)$$

$$\vec{r}\,'(t) = \left\langle \frac{d}{dt}(t^2), \frac{d}{dt}(\sqrt{t-5}) \right\rangle = \left\langle 2t, \frac{1}{2\sqrt{t-5}} \right\rangle$$

$$D(\vec{r}\,'(t)) = (5, +\infty)$$

Remark If $\vec{r}(t) = \langle \sqrt{6-t}, \sqrt{t-5} \rangle$

$$\left.\begin{aligned} &D(\sqrt{6-t}) = (-\infty, 6] \\ &D(\sqrt{5-t}) = (5, +\infty) \end{aligned}\right\} \Rightarrow D(\vec{r}(t)) = [5, 6]$$



DEFINITION 4. *If* $(r)(t) = \langle x(t), y(t)\rangle$ *is a vector function representing the position of a particle at time* t*, then*

- **instantaneous velocity** *at time* t *is* $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$
- **instantaneous speed** *at time* t *is* $|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

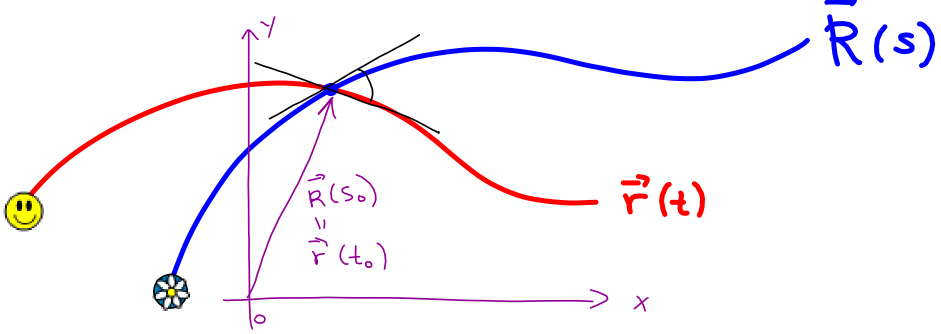
EXAMPLE 5. *The vector function* $\mathbf{r}(t) = \left\langle t, \sqrt{t^2+9}\right\rangle$ *represents the position of a particle at time* t. *Find the velocity and speed of the particle at time* $t = 4$.

$$\vec{r}\,'(t) = \left\langle \frac{d}{dt}(t), \frac{d}{dt}\sqrt{t^2+9} \right\rangle = \left\langle 1, \frac{t}{\sqrt{t^2+9}} \right\rangle$$

$$\vec{r}\,'(4) = \left\langle 1, \frac{4}{\sqrt{4^2+9}} \right\rangle = \left\langle 1, \frac{4}{5} \right\rangle \text{ velocity}$$

$$\text{speed } |\vec{r}\,'(4)| = \sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5}$$

DEFINITION 6. *The* **angle between two intersecting curves** *(curvilinear angle) is defined to be the angle between the tangent lines at the point of intersection.*



EXAMPLE 7. *Given two curves traced by*

$$\mathbf{r}(t) = \langle 1+t, 3+t^2 \rangle, \qquad \mathbf{R}(s) = \langle 2-s, s^2 \rangle.$$

(a) *At what point do the curves intersect?*

$$\vec{r}(t) = \vec{R}(s)$$

$$\langle 1+t,\ 3+t^2 \rangle = \langle 2-s,\ s^2 \rangle$$

$$1+t = 2-s \Rightarrow t = 2-s-1 \Rightarrow t = 1-s$$

$$3+t^2 = s^2$$

$$3+(1-s)^2 = s^2 \Rightarrow 3+1-2s+\cancel{s^2} = \cancel{s^2}$$

$$4-2s=0 \Rightarrow s=2$$

$$t = 1-2 = -1$$

$$\vec{r}(-1) = \langle 0, 4 \rangle$$

$$\vec{R}(2) = \langle 0, 4 \rangle$$

$\boxed{(0,4) \text{ intersection point}}$

(b) *Find the angle between the curves.* = angle between tangent lines

= angle between tangent vectors (velocities)

= angle between $\vec{r}\,'(t)$ and $R'(s)$ at $(0,4)$

= angle between $\vec{r}\,'(-1)$ and $R'(2)$,

$$\vec{r}\,'(t) = \langle 1, 2t \rangle \Rightarrow \vec{r}\,'(-1) = \langle 1, -2 \rangle$$

$$\vec{R}\,'(s) = \langle -1, 2s \rangle \Rightarrow \vec{R}\,'(2) = \langle -1, 4 \rangle$$

$$\theta = \measuredangle\ \vec{r}\,'(-1),\ \vec{R}\,'(2)$$

$$\cos\theta = \frac{\langle 1,-2\rangle \cdot \langle -1,4\rangle}{|\langle 1,-2\rangle| \cdot |\langle -1,4\rangle|} = \frac{-1-8}{\sqrt{5}\sqrt{17}} = -\frac{9}{\sqrt{85}}$$

$$\theta = \arccos\left(-\frac{9}{\sqrt{85}}\right) = \ldots$$