3.8: Higher Derivatives

The derivative of a differentiable function f is also a function and it may have a derivative of its own:

$$(f')' = f'' \qquad \textbf{second derivative}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f'(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$$

`Alternative Notation:` If $y = f(x)$ then

$$y'' = f''(x) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D^2 f(x).$$

Similarly, the **third derivative** $f''' = (f'')'$ or

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = D^3 f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is denoted by $f^{(n)}(x)$:

$$y^{(n)} = f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right) = D^n f(x).$$

EXAMPLE 1. *If* $y = x^5 + 3x + 1$ *find* $y^{(n)}(x)$

$$y = x^5 + 3x + 1$$

$$y' = 5x^4 + 3$$

$$y'' = 20x^3$$

$$y''' = 60x^2$$

$$y^{(4)} = 120x$$

$$y^{(5)} = 120$$

$y^{(6)} = 0$, $y^{(7)} = 0$ ... $y^{(n)} = 0$ for $n = 6, 7, 8, \ldots$

EXAMPLE 2. *Find $D^{2013}\sin x$.*

$$D\sin x = (\sin x)' = \cos x$$

$$D^2\sin x = D(D\sin x) = D(\cos x) = -\sin x$$

$$D^3\sin x = D(-\sin x) = -\cos x$$

$$D^4\sin x = D(-\cos x) = \sin x$$

$$D^5\sin x = D(D^4\sin x) = D\sin x = \cos x$$

cycle

$D^{4k}\sin x = \sin x$

for $k = 1, 2, 3, \dots$

$2013 : 4 = 503$ (Rem. is 1)

$$2013 = 4\cdot 503 + 1$$

$$D^{2012}\sin x = D^{4\cdot 503}\sin x = \sin x$$

$$D^{2013}\sin x = D\left(D^{2012}\sin x\right) = D(\sin x) = \boxed{\cos x}$$

Implicit second derivatives:

EXAMPLE 3. *Find* $y''(x)$ *if* $x^6 + y^6 = 66$.

$$\frac{d}{dx}\left(x^6 + (y(x))^6\right) = \frac{d}{dx}66$$

$$6x^5 + 6(y(x))^5\, y'(x) = 0$$

$$y'(x) = -\frac{x^5}{(y(x))^5}$$

$$y'' = \frac{d}{dx}\left(-\frac{x^5}{(y(x))^5}\right) = -\frac{5x^4(y(x))^5 - x^5 \cdot 5(y(x))^4 y'(x)}{(y(x))^{10}}$$

$$y'' = -\frac{5x^4y^5 - 5x^5y^4 y'}{y^{10}} = -\frac{5x^4y^5 + 5x^5y^4\frac{x^5}{y^5}}{y^{10}}$$

$$y'' = -\frac{5x^4y^5 + 5\frac{x^{10}}{y}}{y^{10}} = -\frac{\frac{5x^4y^6 + 5x^{10}}{y}}{y^{10}}$$

$$y'' = -\frac{5x^4y^6 + 5x^{10}}{y^{11}} = -\frac{5x^4(\overbrace{y^6 + x^6}^{66})}{y^{11}}$$

$$y'' = -\frac{5x^4 \cdot 66}{y^{11}}$$

$$\boxed{y'' = -\frac{330x^4}{y^{11}}}$$