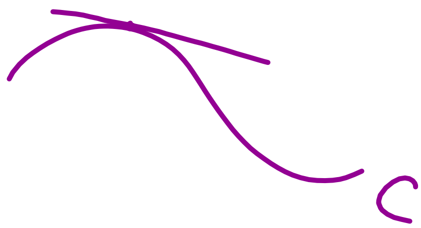
3.9: Slopes and tangents of parametric curves

Consider a curve C given by the parametric equations

$$x = x(t), \quad y = y(t),$$



or in vector form:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle .$$

If both $x(t)$ and $y(t)$ are differentiable, then

$$\mathbf{r}'(t) = \left\langle x'(t), y'(t) \right\rangle$$

is a vector that is tangent to C. Its slope is:

$$\text{slope} = \frac{y'(t)}{x'(t)}$$

Another way to see this is by using the Chain Rule. We have $y = y(x(t))$ and then $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t}$ which implies $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

EXAMPLE 1. *Find the equation of tangent line to the curve*

$$x(t) = \sin t, \quad y(t) = \tan t$$

at the point corresponding to $t = \frac{\pi}{4}$.

Find tangent point at $t = \frac{\pi}{4}$:

$$\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\sin\frac{\pi}{4}, \tan\frac{\pi}{4}\right) = \boxed{\left(\frac{\sqrt{2}}{2}, 1\right)}$$

$$\text{slope} = \frac{y'}{x'} = \frac{\sec^2 t}{\cos t} = \frac{\frac{1}{\cos^2 t}}{\cos t} = \frac{1}{\cos^3 t}$$

$$\text{slope}\Big|_{t=\frac{\pi}{4}} = \frac{1}{\cos^3\frac{\pi}{4}} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3} = (\sqrt{2})^3 = \boxed{2\sqrt{2}}$$

$$\boxed{y - 1 = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right)}$$

EXAMPLE 2. *Find the equation of tangent line to the curve*

$$x(t) = t + 1, \quad y(t) = t^2 + 4$$

at the $(2,5)$ tangent point

$t+1 = 2 \Rightarrow t = 1$

$t^2 + 4 = 5 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$ } $\boxed{t = 1}$

$\text{slope} = \frac{y'}{x'} = \frac{2t}{1} = 2t$

$\boxed{y - 5 = 2(x-2)}$

$\text{slope}\Big|_{t=1} = 2$

EXAMPLE 3. *Find the points on the curve*

$$x = t + t^2, \quad y = t^2 - t$$

$x' = 1+2t \qquad y' = 2t-1$

where the tangent lines are horizontal and there they are vertical.

$\text{slope} = \dfrac{y'}{x'}$

horizontal

$y' = 0 \quad , \quad x' \neq 0$

$2t-1=0 \quad , \quad 1+2t \neq 0$

$t = \frac{1}{2} \qquad t \neq -\frac{1}{2}$

$\left(x(\tfrac{1}{2}), y(\tfrac{1}{2})\right) = \left(\tfrac{3}{4}, -\tfrac{1}{4}\right)$

$x(\tfrac{1}{2}) = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$

$y(\tfrac{1}{2}) = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4}$

vertical

$x' = 0 \quad , \quad y' \neq 0$

$\Longrightarrow \quad t = -\frac{1}{2}$

$\left(x(-\tfrac{1}{2}), y(-\tfrac{1}{2})\right) = \left(-\tfrac{1}{4}, \tfrac{3}{4}\right)$

$x(-\tfrac{1}{2}) = -\tfrac{1}{2} + \tfrac{1}{4} = -\tfrac{1}{4}$

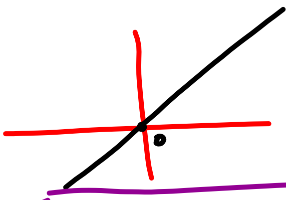
$y(-\tfrac{1}{2}) = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$

REMARK 4. It may happen that $x'(t) = y'(t) = 0$ for some value of t.

Illustration 1. $x(t) = t^3, \; y(t) = t^3$ $\Rightarrow y = x$

$x'(t) = y'(t) = 3t^2 \Rightarrow m = \frac{y'}{x'} = \frac{3t^2}{3t^2} = 1$ if $t \neq 0$

$x'(0) = y'(0) = 0$



One can reparameterize the curve as

$x = t, \; y = t$

then $m = \frac{y'}{x'} = 1$ for all t.

Illustration 2. $x(t) = t^3, \; y(t) = t^2$

$x'(t) = 3t^2$

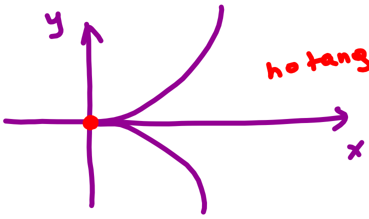
$y'(t) = 2t$ $\Rightarrow x'(0) = y'(0) = 0$

Eliminate parameter

$x = t^3 \Rightarrow x^2 = t^6$

$y = t^2 \Rightarrow y^3 = t^6$ $\Big\} \Rightarrow x^2 = y^3$

$x = \pm\sqrt{y^3}$



no tangent at origin

EXAMPLE 4. *Show that the curve*

$$x = \cos t, \quad y = \cos t \sin t = \frac{1}{2}\sin 2t$$

has two tangents at $\underbrace{(0,0)}$ *and find their equations.*

$\underbrace{\cos t = 0}$ $\Downarrow$

$t = \frac{\pi}{2} + 2\pi k$

OR

$t = \frac{3\pi}{2} + 2\pi k$

where $k = 0, \pm 1, \pm 2, \dots$

$\underbrace{\cos t \sin t = 0}$

$\cos t = 0$ OR $\underbrace{\sin t = 0}$ do not need to consider because it doesn't imply $x(t) = \cos t = 0$

$$\text{slope} = \frac{y'}{x'} = \frac{\frac{d}{dt}\left(\frac{1}{2}\sin 2t\right)}{\frac{d}{dt}(\cos t)} = \frac{\cos 2t}{-\sin t}$$

$$\text{slope}\Big|_{t=\frac{\pi}{2}+2\pi k} = \frac{\cos(\pi + 4\pi k)}{-\sin\left(\frac{\pi}{2}+2\pi k\right)} = \frac{\cos \pi}{-\sin\left(\frac{\pi}{2}\right)} = \frac{-1}{-1} = 1$$

Tangent $\boxed{y = x}$

$$\text{slope}\Big|_{t=\frac{3\pi}{2}+2\pi k} = \frac{\cos(3\pi + 4\pi k)}{-\sin\left(\frac{3\pi}{2}+2\pi k\right)} = \frac{\cos 3\pi}{-\sin\frac{3\pi}{2}} = \frac{-1}{-(-1)} = -1$$

Tangent $\boxed{y = -x}$