4.1: Exponential functions and their derivatives

An **exponential function** is a function of the form

$$f(x) = a^x \quad , a > 0$$

where a is a positive constant. It is defined is the following manner:

- If $x = n$, a positive integer, then $a^n = \underbrace{a \cdot a \cdot \quad \cdots \quad \cdot a}_{n \text{ factors}}$ $\qquad a^3 = a \cdot a \cdot a$
- If $x = 0$ then $a^0 = 1$.
- If $x = -n$, n is a positive integer, then $a^{-n} = \frac{1}{a^n}$. $\qquad a^{-3} = \frac{1}{a^3}$
- If x is a rational number, $x = \frac{p}{q}$, with p and q integers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

$$a^{2/3} = \sqrt[3]{a^2} \qquad a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$$

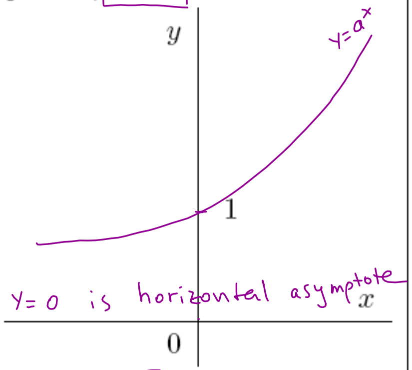
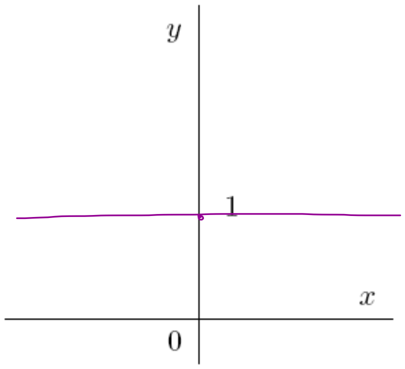
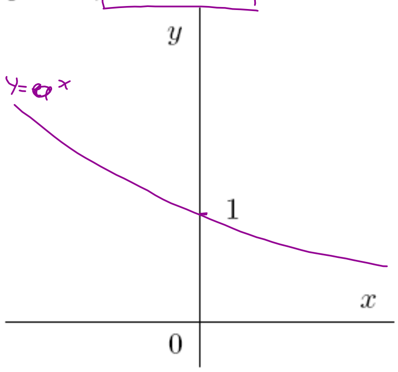
- If x is an irrational number then we define

$$a^x = \lim_{r \to x} a^r$$

where r is a rational number.

It can be shown that this definition uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

There are basically 3 kinds of exponential functions $y = a^x$:

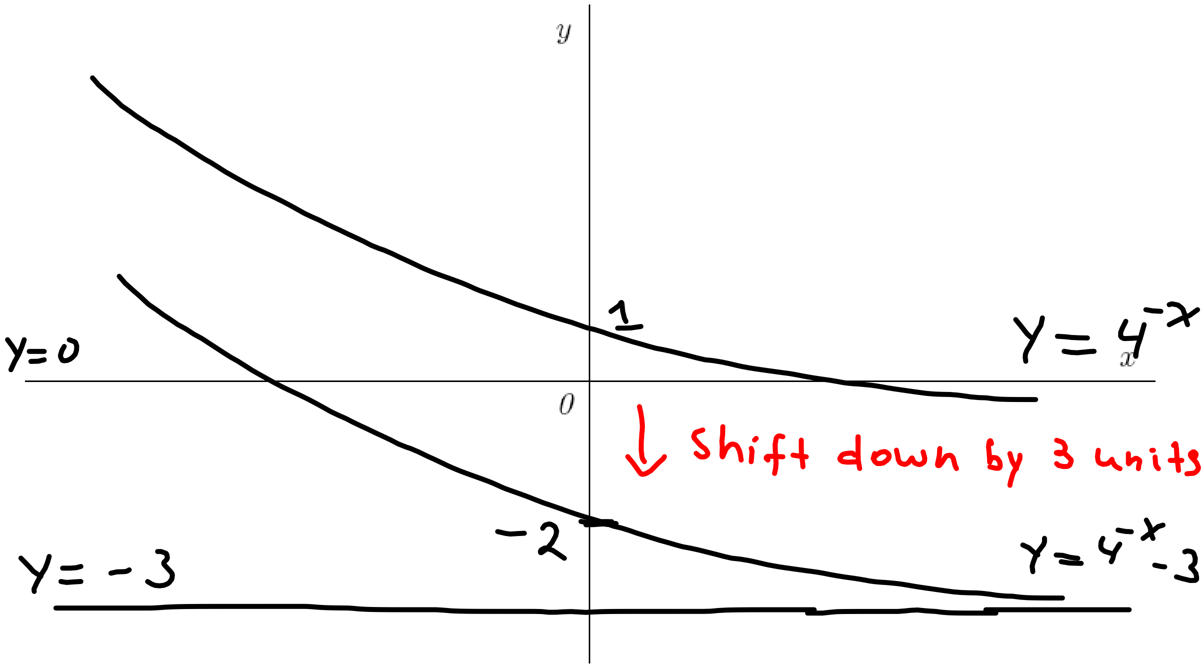
Exponential growth	Constant	Exponential Decay
$y = a^x, \boxed{a > 1}$	$y = 1^x, a = 1$	$y = a^x, \boxed{0 < a < 1}$
		
Domain: $\mathbb{R}$	Domain: $\mathbb{R}$	Domain: $\mathbb{R}$
Range: $(0, \infty)$	Range: $\{1\}$	Range: $(0, \infty)$
$\lim_{x \to \infty} a^x = \infty$		$\lim_{x \to \infty} a^x = 0$
$\lim_{x \to -\infty} a^x = 0$		$\lim_{x \to -\infty} a^x = \infty$

EXAMPLE 1. (a) *Find*

$$\lim_{x\to\infty}(4^{-x}-3) = \lim_{x\to\infty}\left(\frac{1}{4}\right)^x - 3 = 0-3=-3$$

$$0<\frac{1}{4}<1$$

(b) *Sketch the graph of the function* $y = 4^{-x} - 3$ *using transformations of graphs.*



PROPERTIES OF THE EXPONENTIAL FUNCTION: If $a, b > 0$ and x, y are real then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$.

EXAMPLE 2. *Find the limit:*

(a) $\lim\limits_{x\to\infty} \left(\dfrac{\pi}{7}\right)^x = 0$

$0 < \dfrac{\pi}{7} < 1$

(b) $\lim\limits_{x\to-\infty} (\pi^2 - 7)^x = 0$

$\pi^2 - 7 > 1$

(c) $\lim\limits_{x\to 3+} \left(\dfrac{1}{7}\right)^{\frac{x}{x-3}} = \lim\limits_{u\to\infty} \left(\dfrac{1}{7}\right)^u = 0$

$u = \dfrac{x}{x-3} \xrightarrow[x\to 3^+]{} \infty$

$x > 3 \Rightarrow \dfrac{x}{x-3} > 0$

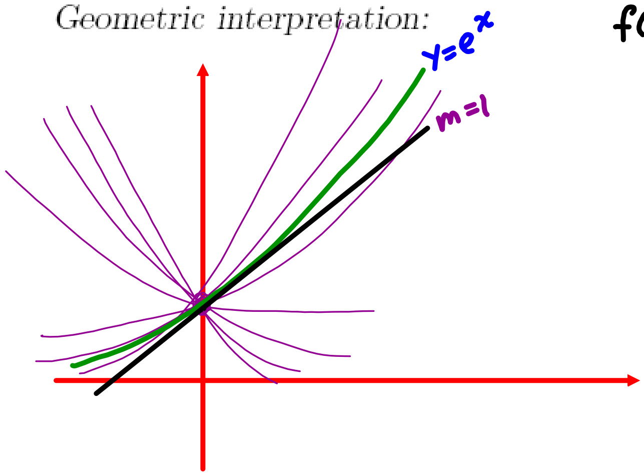
There are in fact a variety of ways to define e. Here are a two of them:

1. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

2. e is the unique positive number for which $\boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1.}$

It can be also shown that $e \approx 2.71828$.

Geometric interpretation:



$f(x) = e^x$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{e^h-e^0}{h} = \lim_{h\to 0}\frac{e^h-1}{h} = 1$$

EXAMPLE 3. *Find the limit:*

(a) $\lim_{x\to 1^+} e^{\frac{4}{x-1}} = \lim_{u\to\infty} e^u = \infty$

$e > 1$

$u = \frac{4}{x-1} \xrightarrow[x\to 1^+]{} \infty$

$\frac{4}{x-1} > 0$ if $x > 1$

(b) $\lim_{x\to 1^-} e^{\frac{4}{x-1}} = \lim_{u\to -\infty} e^u = 0$

where $u = \frac{4}{x-1}$

(c) $\lim_{x\to\infty} \frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}}$

Way 1

$$\frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}} = \frac{e^{5x} - \frac{1}{e^{5x}}}{e^{5x} + \frac{1}{e^{5x}}} = \frac{\frac{e^{5x}\cdot e^{5x} - 1}{\cancel{e^{5x}}}}{\frac{e^{5x}e^{5x} + 1}{\cancel{e^{5x}}}} = \frac{e^{10x} - 1}{e^{10x} + 1} = \ldots$$

Way 2

$$\frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}} = \frac{u - \frac{1}{u}}{u + \frac{1}{u}} = \frac{(u^2-1)/u}{(u^2+1)/u} = \frac{u^2-1}{u^2+1} \xrightarrow[u\to\infty]{} \frac{1}{1} = \boxed{1}$$

$\boxed{u = e^{5x} \xrightarrow[x\to\infty]{} \infty}$

Derivative of exponential function.

EXAMPLE 4. *Find the derivative of* $f(x) = e^x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h-1)}{h} = e^x \underbrace{\lim_{h\to 0} \frac{e^h-1}{h}}_{=1} = e^x \qquad (e^x)' = e^x$$

CONCLUSIONS:

1. e^x is differentiable function.

2. If $u(x)$ is a differentiable function then by Chain Rule: $\dfrac{\mathrm{d}}{\mathrm{d}x} e^{u(x)} = e^u \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

EXAMPLE 5. *Find y'' for e^{x^2}.*

$$y' = \left(e^{x^2}\right)' \overset{\text{C.R.}}{=} e^{x^2}\,(x^2)' = \underbrace{2x\,e^{x^2}}$$

$$y'' = \left(2x\,e^{x^2}\right)' = 2\left(x e^{x^2}\right)' \overset{\text{P.R.}}{=} 2\left[e^{x^2} + x \cdot 2x e^{x^2}\right]$$

$$= 2e^{x^2}\left(1 + 2x^2\right)$$

EXAMPLE 6. *Find the derivative:*

(a) $y = \sqrt{e^x + x^3}$

$$y' = \frac{1}{2\sqrt{e^x + x^3}} \cdot \frac{d}{dx}\left(e^x + x^3\right)$$

$$= \frac{e^x + 3x^2}{2\sqrt{e^x + x^3}}$$

(b) $y = e^{x \sin x}$

$$y' = e^{x \sin x}\left(x \sin x\right)' \overset{P.R.}{=}$$

$$e^{x \sin x}\left(\sin x + x \cos x\right)$$

EXAMPLE 7. *For what value(s) of A does the function $y = e^{Ax}$ satisfy the equation $y'' + 2y' - 8y = 0$?*

$$y'' = A^2 e^{Ax}$$
$$+\ 2y' = 2A e^{Ax}$$
$$+\ -8y = -8e^{Ax}$$
$$0 = A^2 e^{Ax} + 2A e^{Ax} - 8 e^{Ax}$$

$$0 = \underbrace{e^{Ax}}_{\neq 0} (A^2 + 2A - 8)$$

$$\Rightarrow \quad A^2 + 2A - 8 = 0$$
$$(A+4)(A-2) = 0$$
$$\boxed{A = -4, \quad A = 2}$$

Conclusion: The functions $y = e^{-4x}$ and $y = e^{2x}$ are solutions of the differential equation

$$y'' + 2y' - 8y = 0.$$