4.2:Inverse Functions

DEFINITION 1. *A function of domain X is said to be a* **one-to-one** *function if no two elements of X have the same image, i.e.*

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

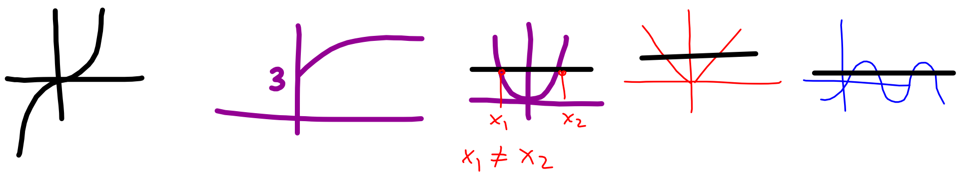
Horizontal line test:A function if one-to-one is and only if no horizontal line intersects its graph more once.

EXAMPLE 2. *Are the following functions one-to-one?*

1-1   1-1   not 1-1   not 1-1   not 1-1   not 1-1

$$f(x) = x^3, \quad g(x) = \sqrt{x} + 3, \quad h(x) = x^2, \quad u(x) = |x|, \quad w(x) = \sin x, \quad F(x) = -x^2 + x + 1$$

parabola



$x_1 \neq x_2$
but $f(x_1) = f(x_2)$

EXAMPLE 3. *Prove that* $f(x) = \dfrac{x-3}{x+3}$ *is one-to-one.* on its domain
$D(f) = \{x \mid x \neq -3\}$

If $\boxed{f(x_1) = f(x_2)}$ then show that $\boxed{x_1 = x_2.}$

$$\Downarrow$$

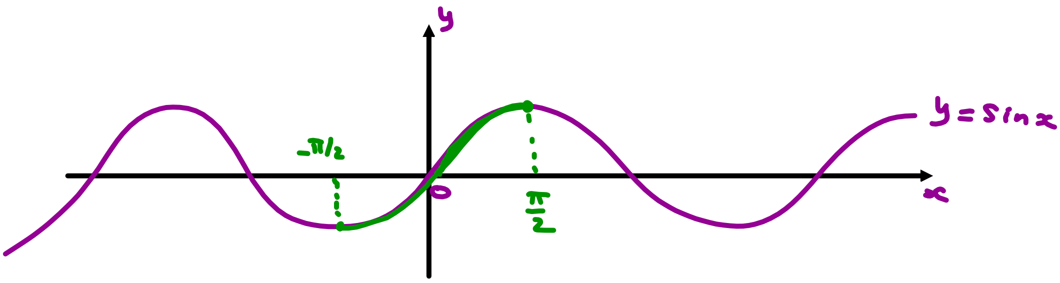
$$\frac{x_1-3}{x_1+3} = \frac{x_2-3}{x_2+3}$$

$$(x_1-3)(x_2+3) = (x_2-3)(x_1+3)$$

$$\cancel{x_1 x_2} - \cancel{9} - 3x_2 + 3x_1 = \cancel{x_2 x_1} - \cancel{9} - 3x_1 + 3x_2$$

$$6x - 6x_2 = 0 \implies \boxed{x_1 = x_2}$$

EXAMPLE 4. *How we can restrict the domain of $f(x) = \sin x$ to make it one-to-one?*



F. ex. $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

DEFINITION 5. *Let f be a one-to-one function with domain X and range Y. Then the inverse function f^{-1} has the domain Y and range X and is defined for any y in Y by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y.$$

REMARK 6. Reversing roles of x and y in the last formula we get:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

REMARK 7. If $y = f(x)$ is one-to-one function with the domain X and the range Y then

for every x in X $\quad f^{-1}(f(x)) = f^{-1}(y) = x$ $\qquad f^{-1} \circ f = f \circ f^{-1}$

and

for every x in Y $\quad f(f^{-1}(x)) = f(y) = x$

CAUTION: $f^{-1}(x)$ **does NOT mean** $\dfrac{1}{f(x)}$.

TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f:

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 8. *(cf. Example3) Find the inverse function of* $f(x) = \dfrac{x-3}{x+3}$.

$$y = \frac{x-3}{x+3}$$

$$(x+3)\,y = x-3 \Rightarrow xy + 3y = x-3$$

$$xy - x = -3 - 3y$$

$$x(y-1) = -3(1+y)$$

$$x = \frac{-3(1+y)}{y-1}$$

$$y = \boxed{-\frac{3(1+x)}{x-1} = f^{-1}(x)}$$

In addition, let us find domain and range for f, f^{-1}

	Domain	Range
$f(x)$	$x \neq -3$	$y \neq 1$
$f^{-1}(x)$	$x \neq 1$	$y \neq -3$

EXAMPLE 9. *Given* $f(x) = x^2 + x$, $x \geq -\frac{1}{2}$. *Find the inverse function of* f.

$$y = x^2 + x$$

$$x^2 + x - y = 0$$

Use $ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have

$a = 1$, $b = 1$, $c = -y$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 4y}}{2}$$

$$x = \frac{-1 + \sqrt{1 + 4y}}{2}$$

$$y = \boxed{\frac{-1 + \sqrt{1 + 4x}}{2} = f^{-1}(x)}$$

In addition, determine domain and range of f, f^{-1}.

$f^{-1}(x)$ is defined for all x s.t.

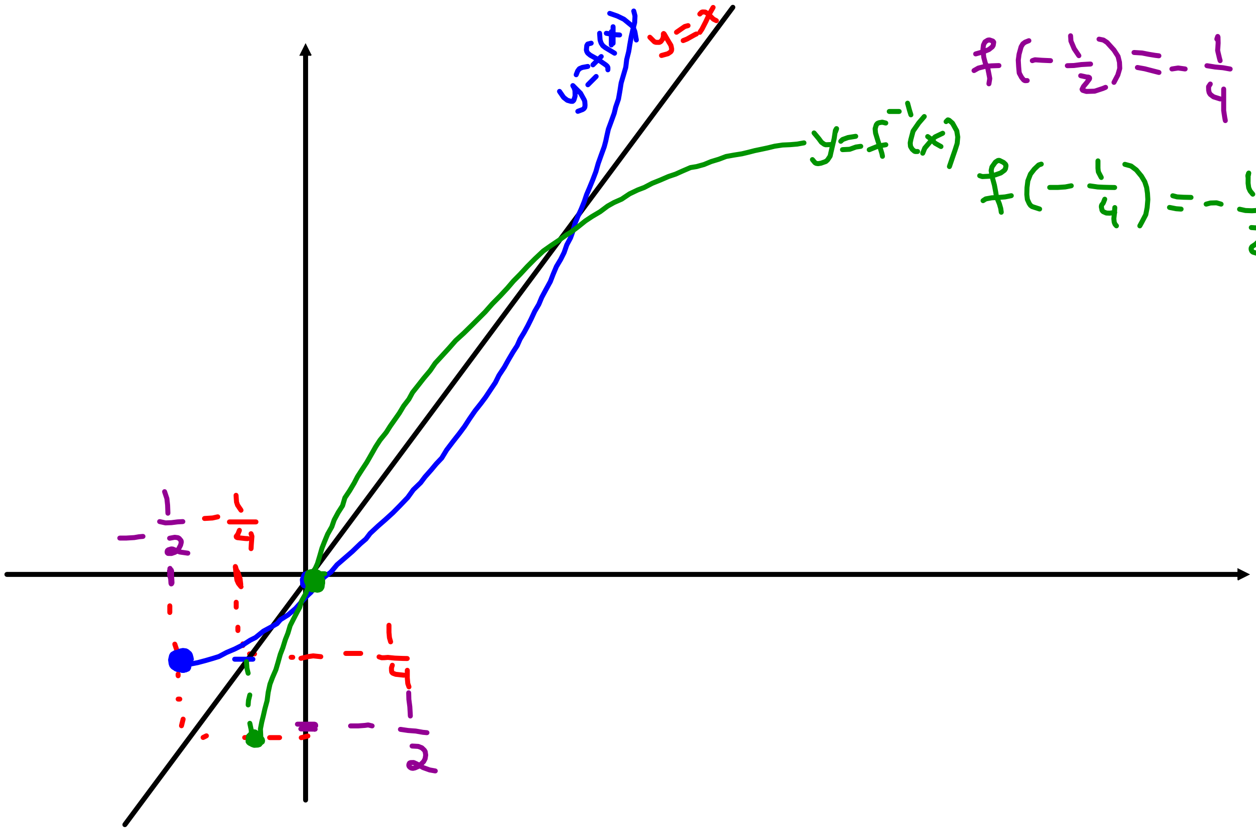
$$1 + 4x \geq 0$$

$$x \geq -\frac{1}{4}$$

	Domain	Range
f	$x \geq -\frac{1}{2}$	$y \geq -\frac{1}{4}$
f^{-1}	$x \geq -\frac{1}{4}$	$y \geq -\frac{1}{2}$

FACT: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Graph $y=f(x)$ and $y=f^{-1}(x)$ for the function from ex. 9.



$f(-\frac{1}{2}) = -\frac{1}{4}$

$f(-\frac{1}{4}) = -\frac{1}{2}$

If (a, b) belongs to the graph of $y=f(x)$
then (b, a) belongs to the graph of $y=f^{-1}(x)$

THEOREM 10. *If f is a one-to-one differentiable function with inverse function* $\boxed{g = f^{-1}}$ *and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and*

$$g'(a) = \frac{1}{f'(g(a))}.$$

Proof. $f(f^{-1}(x)) = x$

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}(x)$$

Use Chain Rule: $f'(g(x))\,g'(x) = 1$

$$g'(x) = \frac{1}{f'(g(x))} \Rightarrow g'(a) = \frac{1}{f'(g(a))}.$$

EXAMPLE 11. *Suppose that* $\underbrace{\text{$g$ is the inverse function of f}}$ $g = f^{-1}$ *and* $f(4) = 5$, $f'(4) = 7$. *Find* $g'(5)$.

$$g'(5) = \frac{1}{f'(g(5))}$$

$$\boxed{4 = f^{-1}(5) = g(5)}$$

$$g'(5) = \frac{1}{f'(4)} = \boxed{\frac{1}{7}}$$

EXAMPLE 12. *Suppose that g is inverse of f. Find $g'(a)$ where*

(a) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$

$g = f^{-1}$ Find $g'(2)$.

$g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$

$\boxed{g'(2) = \frac{2}{3}}$

$g(2) = x$

or $f^{-1}(2) = x \Rightarrow f(x) = 2$

Guess $x = 1 \Rightarrow f(1) = \sqrt{4} = 2$

So, $\boxed{g(2) = 1}$

Find $f'(1)$:

$f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}} \Rightarrow f'(1) = \dfrac{3+2+1}{2\sqrt{4}} = \dfrac{6}{4} = \boxed{\dfrac{3}{2}}$

(b) $f(x) = \dfrac{2x-3}{x+3}$, $a = \dfrac{1}{2}$.

$g = f^{-1}$ Find $g'\left(\frac{1}{2}\right)$

$$g\left(\tfrac{1}{2}\right) = x$$

$$f^{-1}\left(\tfrac{1}{2}\right) = x$$

Find x such that

$$\tfrac{1}{2} = f(x)$$

$$\frac{1}{2} = \frac{2x-3}{x+3}$$

$$x+3 = 2(2x-3)$$

$$x+3 = 4x-6$$

$$3x = 9$$

$$\boxed{x=3} \Rightarrow \boxed{g\left(\tfrac{1}{2}\right)=3}$$

$$g'\left(\tfrac{1}{2}\right) = \frac{1}{f'\left(g\left(\frac{1}{2}\right)\right)} = \frac{1}{f'(3)} = \frac{1}{\frac{1}{4}} = 4$$

Quotient Rule

$$f'(x) = \frac{d}{dx}\left(\frac{2x-3}{x+3}\right) = \frac{2(x+3) - (2x-3)}{(x+3)^2}$$

$$f'(3) = \frac{2\cdot 6 - 3}{6^2} = \frac{9}{36} = \frac{1}{4}$$

$$\boxed{g'\left(\tfrac{1}{2}\right) = 4}$$

Find $g'(a)$ where $g = f^{-1}$

(c) $f(x) = 4 + 3x + e^{3(x-1)}$, $a = 8$.

$$g'(a) = \frac{1}{f'(g(a))}$$

$$g'(8) = \frac{1}{f'(g(8))}$$

$$g'(8) = \frac{1}{f'(1)}$$

$$\boxed{g'(8) = \frac{1}{6}}$$

Find $g(8)$:

$g(8) = x$

$f^{-1}(8) = x$

$8 = f(x)$

Find x s.t. $f(x) = 8$

Guess $\underset{x=1}{f(1) = 4 + 3 + e^0} = 8$

So, $\boxed{g(8) = 1}$

Find $f'(1)$:

$f'(x) = 0 + 3 + 3e^{3(x-1)}$

$f'(1) = 3 + 3 \cdot 1 = 6$