4.3:Logarithmic Functions

DEFINITION 1. *The exponential function $f(x) = a^x$ with $a \neq 1$ is a one-to-one function. The inverse of this function, called the* **logarithmic function** *with* **base** *a, is denoted by $f^{-1}(x) = \log_a x$.*

Namely,

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x. > 0$$

In other words, if $x > 0$ then $\log_a(x)$ is the exponent to which the base a must be raised to give x.

EXAMPLE 2. *Evaluate*

(a) $\log_2 16 = \log_2 2^4 = 4$

(b) $\log_3 \frac{1}{81} = \log_3 3^{-4} = -4$

(c) $\log_{125} 5 = \log_{125} 125^{\frac{1}{3}} = \frac{1}{3}$

$2^{\square} = 16$

$\frac{1}{81} = \frac{1}{3^4} = 3^{-4}$

$5 = \sqrt[3]{125} = 125^{\frac{1}{3}}$

(d) $\log_{125} 1 = 0$

(e) $\log_a 1 = 0$ because $a^0 = 1$

$f(x) = a^x \Rightarrow f^{-1}(x) = \log_a x$

CANCELLATION RULES:

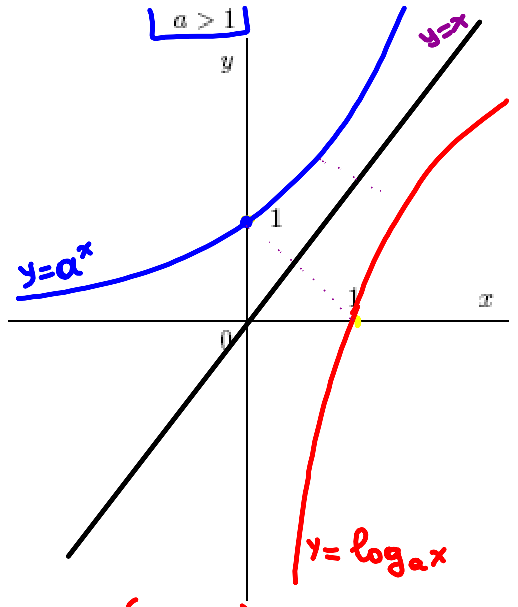
$f^{-1}(f(x)) = x$

- $\log_a a^x = x$ for all $x \in \mathbb{R}$

$f(f^{-1}(x)) = x$

- $a^{\log_a x} = x$ for $x > 0$.

Graphs of logarithmic functions $y = \log_a x$:



Domain: $(0, +\infty)$

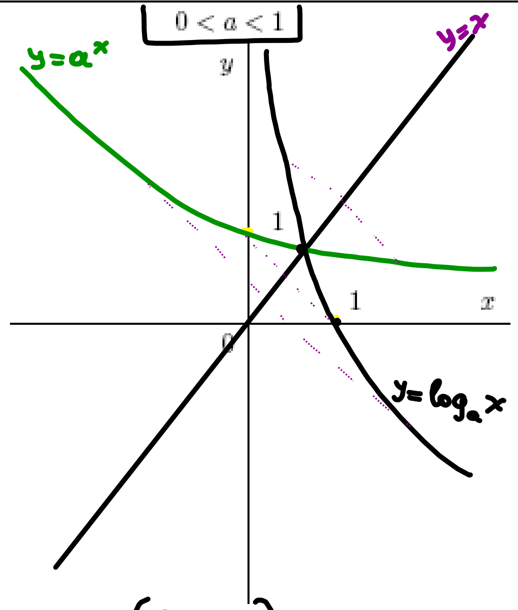
Range: $\mathbb{R}$

$\lim_{x\to\infty} \log_a x = \infty$

$\lim_{x\to 0^+} \log_a x = -\infty$

Vertical asymptote: $x = 0$

Horizontal asymptote: NO



Domain: $(0, \infty)$

Range: $\mathbb{R}$

$\lim_{x\to\infty} \log_a x = -\infty$

$\lim_{x\to 0^+} \log_a x = +\infty$

Vertical asymptote: $x = 0$

Horizontal asymptote: NO

In particular,

$\log_a \sqrt{x} = \log_a x^{\frac{1}{2}}$

$= \frac{1}{2} \log_a x$

Properties: Assume that $a \neq 1$ and $x, y > 0$.

$$(1) \quad \log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_b y$$

$$\log_a(x^y) = y \log_a x$$

Notation: *Common Logarithm:* $\log x = \log_{10} x$. (Thus, $\log x = y \quad \Leftrightarrow \quad 10^y = x$.)
Natural Logarithm: $\ln(x) = \log_e(x)$. (Thus, $\ln x = y \quad \Leftrightarrow \quad e^y = x$.)

Proof of (1):

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$a^{\log_a(xy)} = a^{\log_a(x) + \log_a(y)}$$

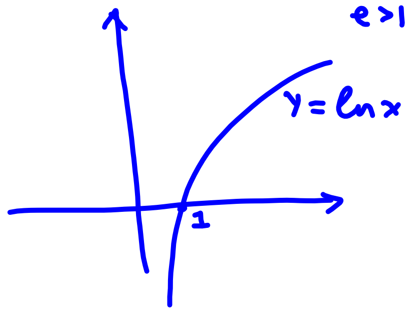
Cancellation Rule

$$xy = a^{\log_a(x)} \cdot a^{\log_a(y)}$$

$$xy = x \cdot y \quad \text{(TRUE)}.$$

Properties of the natural logarithms:

- $\ln(e^x) = \log_e(e^x) = x$
- $e^{\ln x} = e^{\log_e(x)} = x \quad , x > 0$
- $\ln e = \log_e e = 1$
- $\boxed{\log_a x = \dfrac{\ln x}{\ln a}, \text{ where } a > 0 \text{ and } a \neq 1;}$
- $\lim_{x\to\infty} \ln x = \infty$
- $\lim_{x\to 0+} \ln x = -\infty$



Proof:

$$\ln a \log_a(x) = \ln x$$

$$e^{\ln a \log_a(x)} = e^{\ln x}$$

$$\left(\underbrace{e^{\ln a}}\right)^{\log_a(x)} = x$$

$$a^{\log_a(x)} = x$$

$$x = x \quad \text{(TRUE)}$$

EXAMPLE 3. *Find each limit:*

(a) $\lim_{x\to\infty} \ln(x^2 - x) = \lim_{u\to\infty} \ln u = \infty$

$\boxed{u = x^2 - x \xrightarrow[x\to\infty]{} \infty}$

(b) $\lim_{x\to 13+} \log_{13}(x-13) = \lim_{u\to 0^+} \log_{13} u = -\infty$

$u = x-13 \xrightarrow[x\to 13^+]{} 0^+$

$x \to 13^+ \Rightarrow x > 13$

$x - 13 > 0$

(c) $\lim_{x\to 0+} \log(\sin x) = \lim_{u\to 0^+} \log(u) = -\infty$

$\boxed{u = \sin x \xrightarrow[x\to 0^+]{} 0^+}$

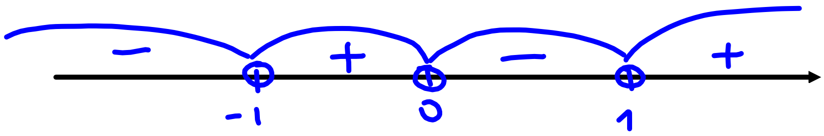
EXAMPLE 4. *Find the domain of* $f(x) = \ln(x^3 - x)$.

$$\left.\begin{array}{l} D(\ln u) = \{u \mid u > 0\} \\ x^3 - x \text{ is defined for all } x \end{array}\right\} \Rightarrow$$

$$x^3 - x > 0$$

$$x(x^2 - 1) > 0$$

$$x(x-1)(x+1) > 0$$



$$D(\ln(x^3 - x)) = (-1, 0) \cup (1, +\infty)$$

EXAMPLE 5. *Solve the following equations:*

(a) $\log_{0.5}(\log(x+120)) = -1$

$x+120>0$

$x+120>1$

$$\log(x+120) = (0.5)^{-1} = \left(\tfrac{1}{2}\right)^{-1} = 2$$

verify

$$\log_{10}(x+120) = 2$$

$$x + 120 = 10^2$$

$$x = 100 - 120$$

$$\boxed{x = -20}$$

(b) $e^{5+2x} = 4$

can be omitted

$$\ln(e^{5+2x}) = \ln 4$$

$$(5+2x)\underbrace{\ln e}_{1} = \ln 4$$

$$5 + 2x = \ln 4$$

$$2x = \ln 4 - 5$$

$$\boxed{x = \frac{1}{2}\ln 4 - \frac{5}{2}}$$

OR

$$\frac{1}{2}\ln 4 = \ln 4^{\frac{1}{2}} = \ln 2$$

$$x = \ln 2 - \frac{5}{2}$$

(c) $\log(x-1)+\log(x+1)=\log 15$

$$\log((x-1)(x+1)) = \log 15$$

$$\log(x^2-1) = \log 15$$

$$x^2-1 = 15$$

$$x^2 = 16$$

$$x = \pm 4$$

The final answer is $\boxed{x=4}$

because $x=-4$ doesn't belong

to domains of $y=\log(x-1)$ and $y=\log(x+1)$

(d) $\ln x^2 - 2\ln\sqrt{x^2+1} = 1$

$$\ln x^2 - \ln\left(\sqrt{x^2+1}\right)^2 = 1$$

$$\ln x^2 - \ln(x^2+1) = 1$$

$$\ln \frac{x^2}{x^2+1} = 1 = \ln e$$

$$\frac{x^2}{x^2+1} = e$$

$$x^2 = e(x^2+1)$$

$$x^2 = ex^2 + e$$

$$x^2 - ex^2 = e$$

$$x^2(1-e) = e$$

$$x^2 = \frac{e}{1-e} < 0$$

$\Rightarrow$ no solution

EXAMPLE 6. *Find the inverse of the function:*

(a) $f(x) = \ln(x + 12)$

$$y = \ln(x+12)$$

$$e^y = x+12$$

$$x = e^y - 12$$

$$y = \boxed{e^x - 12 = f^{-1}(x)}$$

In addition

	domain	Range
f	$x > -12$	$\mathbb{R}$
f^{-1}	$\mathbb{R}$	$y > -12$

(b) $f(x) = \dfrac{10^x - 1}{10^x + 1}$

$$y = \frac{10^x - 1}{10^x + 1}$$

$$(10^x + 1)y = 10^x - 1$$

$$10^x y + y = 10^x - 1$$

$$10^x y - 10^x = -1 - y$$

$$10^x(y-1) = -(1+y)$$

$$10^x = -\frac{1+y}{y-1}$$

$$10^x = \frac{1+y}{1-y}$$

$$x = \log_{10} \frac{1+y}{1-y}$$

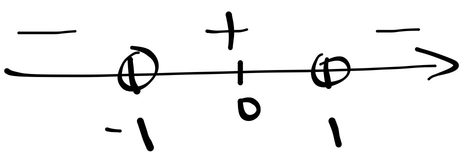
$$y = \boxed{\log_{10} \frac{1+x}{1-x} = f^{-1}(x)}$$

In addition,

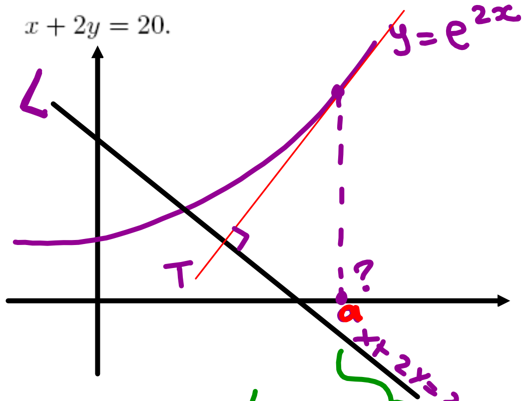
	Domain	Range
f	$\mathbb{R}$	$(-1, 1)$
f^{-1}	$(-1, 1)$	$\mathbb{R}$

$$y = \log_{10} \frac{1+x}{1-x}$$

$$\frac{1+x}{1-x} > 0$$



EXAMPLE 7. *Find an equation of the tangent to the curve* $y = e^{2x}$ *that is perpendicular to the line* $x + 2y = 20$.



$\Downarrow$

$y = \frac{20-x}{2}$ $\Longrightarrow m_L = -\frac{1}{2}$

slope of T at $x=a$

$$m_T = y'(a) = (e^{2x})'\Big|_{x=a} = \underline{2e^{2a}}$$

On the other hand, $T \perp L$, i.e

$$m_T \cdot m_L = -1$$

$$2e^{2a} \cdot \left(-\frac{1}{2}\right) = -1$$

$$-e^{2a} = -1$$

$$e^{2a} = 1$$

$$2a = 0 \Rightarrow \boxed{a=0}$$

$\Downarrow$

$y(a) = y(0) = e^{2\cdot 0} = 1$

Tangent point $(0,1)$

$$m_T = 2e^{2\cdot 0} = \boxed{2}$$

OR $m_T = -\frac{1}{m_L} = -\frac{1}{-1/2} = 2$

Tangent line at $(0,1)$:

$$y - 1 = 2(x - 0)$$

$$\boxed{y = 2x + 1}$$

Change of Base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}.$$

EXAMPLE 8. *Using calculator evaluate* $\log_2 15$ *to* 4 *decimal places.*

$$\log_2 15 = \frac{\ln 15}{\ln 2} \approx 3.9069$$