4.4:Derivatives of Logarithmic Functions

EXAMPLE 1. *Using Implicit Differentiation find the derivatives of the following function:*

(a) $f(x) = \ln x$

$$y = \ln x$$

$$x = e^{y(x)}$$

Use chain Rule here

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(e^{y(x)}\right)$$

$$1 = e^{y(x)}\, y'(x)$$

$$y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{x} \Rightarrow \boxed{(\ln x)' = \frac{1}{x}}$$

(b) $f(x) = a^x$

$y = a^x$

$x = \log_a y$

Use Change base formula:

$$x = \frac{\ln y}{\ln a}$$

$$\frac{d}{dx}(x \ln a) = \frac{d}{dx}(\ln \overbrace{y(x)}^{\text{inner}})$$

Apply Chain Rule

$$\ln a = \frac{1}{y(x)} \cdot y'(x)$$

$$y'(x) = y(x) \ln a$$

$$y'(x) = a^x \ln a$$

$$\boxed{(a^x)' = a^x \ln a}$$

Combining the formulas obtained in Example 1 and Chain Rule one can get

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}a^{g(x)} = a^{g(x)} \ln a \; g'(x)$$

EXAMPLE 2. *Find the derivative:*

(a) $f(x) = \underbrace{\ln}_{\text{outer}}(\overbrace{\sin x}^{\text{inner}})$

$$f'(x) = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x} = \cot x$$

(b) $f(x) = \underbrace{\ln}_{\text{outer}}(\overbrace{1 + \ln(1 + \ln x)}^{\text{inner}})$

$$f'(x) = \frac{1}{1+\ln(1+\ln x)} \cdot \frac{d}{dx}\left(1 + \underbrace{\ln}_{\text{outer}}(\overbrace{1+\ln x}^{\text{inner}})\right)$$

$$= \frac{1}{1+\ln(1+\ln x)}\left(0 + \frac{1}{1+\ln x}\frac{d}{dx}(1+\ln x)\right)$$

$$= \frac{1}{1+\ln(1+\ln x)} \cdot \frac{1}{1+\ln x} \cdot \frac{1}{x}$$

(c) $f(x) = \ln|x|$, $x \neq 0$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$f'(x) = (\ln|x|)' = \begin{cases} \frac{d}{dx}(\ln x), & x > 0 \\ \frac{d}{dx}(\ln(-x)), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1), & x < 0 \end{cases} = \frac{1}{x}$$

$$(\ln|x|)' = \frac{1}{x}$$

(d) $f(x) = \cot^3 x + 3^{\cot x}$

$$f(x) = (\cot x)^3 + 3^{\cot x}$$

$$f'(x) = 3\cot^2 x \frac{d}{dx}(\cot x) + 3^{\cot x} \ln 3 \frac{d}{dx}(\cot x)$$

$$f'(x) = \left(3\cot^2 x + 3^{\cot x} \ln 3\right) \frac{d}{dx}(\cot x)$$

$$f'(x) = \left(3\cot^2 x + 3^{\cot x} \ln 3\right)(-\csc^2 x)$$

EXAMPLE 3. *Using the change of base formula, find the derivative formula for $f(x) = \log_a x$ and generalize it using the Chain Rule.*

$$f(x) = \log_a x = \frac{\ln x}{\ln a}$$

$$f'(x) = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x} \Rightarrow \boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

$$\boxed{\left(\log_a g(x)\right)' = \frac{g'(x)}{g(x) \ln a}}$$

EXAMPLE 4. *Find the derivative of $f(x) = \log_2(\underbrace{3 + x^2 + x^3}_{g(x)})$*

$$f'(x) = \frac{2x + 3x^2}{(3 + x^2 + x^3) \ln 2}$$

Logarithmic Differentiation can be used to find derivative of complicated functions involving products, quotients or powers.

STEPS IN LOGARITHMIC DIFFERENTIATION:

1. Take logarithms of both sides of an equation $y = f(x)$ and simplify (f.ex. split a product or quotient, etc.).
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.
4. Plug in $y = f(x)$.

EXAMPLE 5. *Find the derivative:* $y = (\cos x)^{\sin x}$

$$y = (\cos x)^{\sin x}$$

$$\ln a^b = b \ln a$$

$$\ln y = \ln (\cos x)^{\sin x}$$

$$\ln y(x) = \sin x \ \ln(\cos x)$$

Differentiate implicitely

Product Rule

$$\frac{y'(x)}{y(x)} = (\sin x)' \ln(\cos x) + \sin x \cdot (\ln(\cos x))'$$

$$\frac{y'(x)}{y(x)} = \cos x \ \ln(\cos x) + \sin x \ \frac{-\sin x}{\cos x}$$

$$\frac{y'(x)}{y(x)} = \cos x \ \ln(\cos x) - \frac{\sin^2 x}{\cos x}$$

$$y'(x) = y(x) \left[\cos x \ \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

$$y(x) = (\cos x)^{\sin x}$$

$$y'(x) = (\cos x)^{\sin x} \left[\cos x \ \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$