4.6: Inverse trigonometric functions

- **INVERSE SINE:** If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $f(x) = \sin x$ is one-to-one, thus the inverse exists, denoted by $\sin^{-1}(x)$ or $\arcsin x$.

	$y = \sin x$	$y = \arcsin x$ $= \sin^{-1}(x)$
Domain	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
Range	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Cancellation equations: $f^{-1}(f(x)) = f(f^{-1}(x)) = x$

$$\arcsin(\sin x) = x \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

and

$$\sin(\arcsin x) = x \quad \text{if} \quad -1 \le x \le 1.$$

EXAMPLE 1. *Find the exact values of the expression:*

(a) $\sin^{-1} 0 = 0$, because $\sin 0 = 0$

(b) $\arcsin(-1) = -\frac{\pi}{2}$

(c) $\sin^{-1}(0.5) = \arcsin \frac{1}{2} = \frac{\pi}{6}$

$\sin \frac{\pi}{6} = \frac{1}{2}$

(d) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

$-1 \le \frac{2}{7} \le 1 \Rightarrow$ By Cancellation Rule

(e) $\sin\left(\arcsin\frac{2}{7}\right) = \frac{2}{7}$

Way 1 $= \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

Way 2 using Cancellation Rule

(g) $\arcsin\left(\sin\frac{5\pi}{4}\right)$

$\sin\left(\frac{5\pi}{4}\right) = \sin(-\pi/4)$

$-\frac{\pi}{2} \le -\pi/4 \le \pi/2$

By Cancellation Rule

$\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(\sin\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

(h) $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

(i) $\arcsin\left(\sin\frac{\pi}{150}\right) = \frac{\pi}{150}$

By Cancellation Rule because

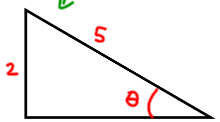
$-\frac{\pi}{2} \le -\pi/3 \le \frac{\pi}{2}$

$-\pi/2 \le \frac{\pi}{150} \le \pi/2$

(f) $\tan\left(\underbrace{\arcsin\frac{2}{5}}_{\theta}\right) = \tan\theta = \frac{2}{\sqrt{21}}$

$\theta = \arcsin \frac{2}{5}$

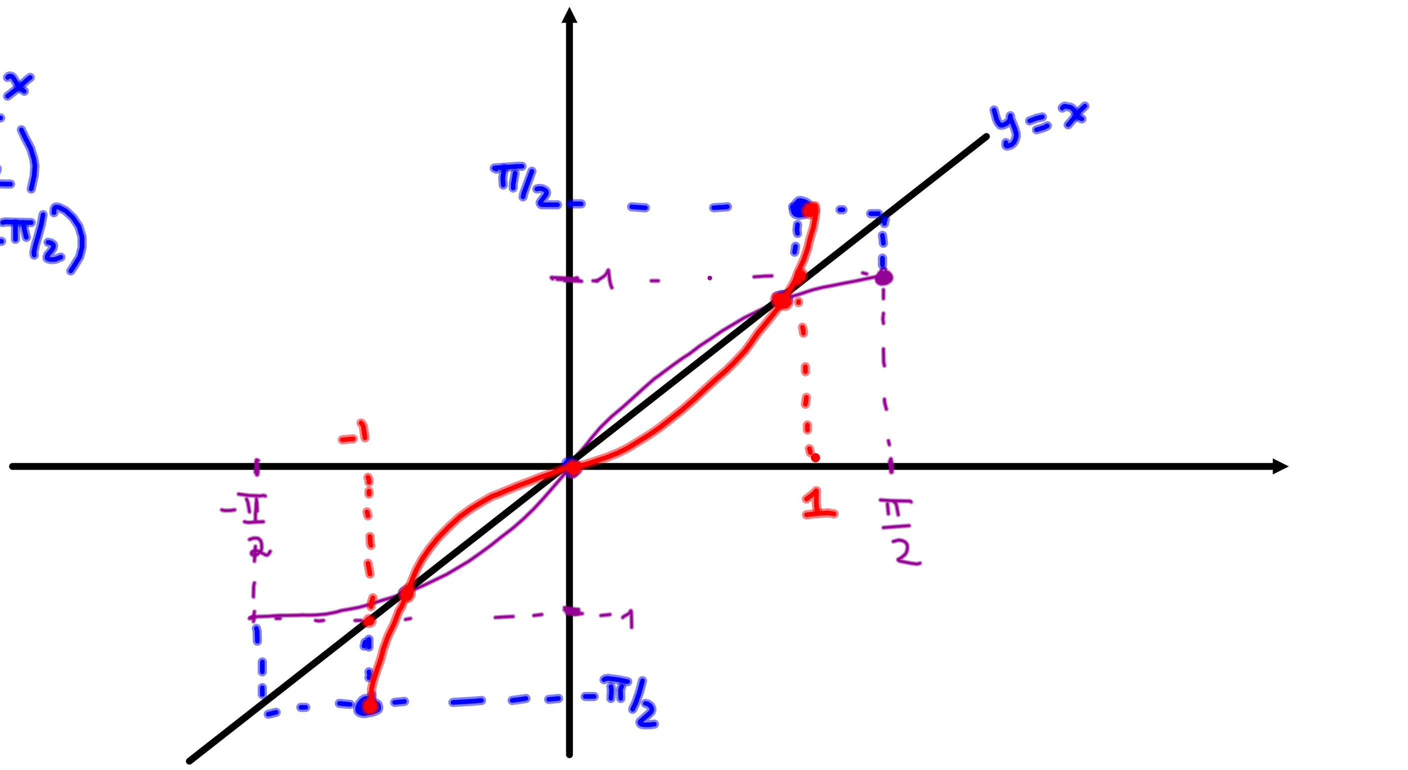
$\sin\theta = \frac{2}{5}$



$\sqrt{5^2 - 2^2} = \sqrt{21}$

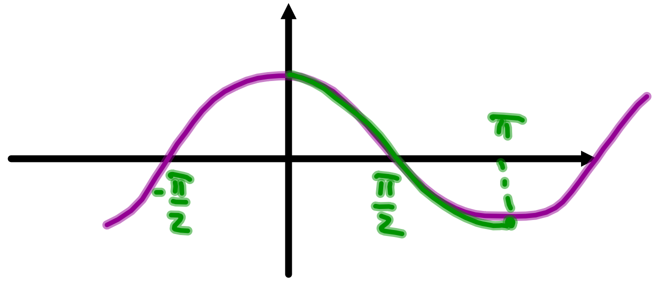
EXAMPLE 2. *Sketch the graph of* $\arcsin(x)$.

$\sin x$	$\arcsin x$
$(\frac{\pi}{2}, 1)$	$(1, \pi/2)$
$(-\frac{\pi}{2}, -1)$	$(-1, -\pi/2)$



- **INVERSE COSINE:** If $0 \leq x \leq \pi$, then $f(x) = \cos x$ is one-to-one, thus the inverse exists, denoted by $\cos^{-1}(x)$ or $\arccos x$.

	$y = \cos x$	$y = \arccos x$ $= \cos^{-1}(x)$
Domain	$[0, \pi]$	$[-1, 1]$
Range	$[-1, 1]$	$[0, \pi]$



Cancellation equations:

$$\arccos(\cos x) = x \quad \text{if} \quad 0 \leq x \leq \pi$$

and

$$\cos(\arccos x) = x \quad \text{if} \quad -1 \leq x \leq 1.$$

Covered during the recitation.

EXAMPLE 3. *Find the exact values of the expression:*

(a) $\arccos 0$

(b) $\cos^{-1} 1$

(c) $\arccos(-1)$

(d) $\arccos 0.5$

(e) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

(f) $\sin\left(2\arccos\frac{3}{5}\right)$

(g) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

(h) $\arccos\left(\cos\dfrac{7\pi}{6}\right)$

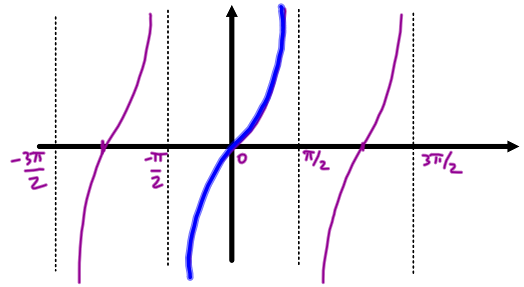
(i) $\cos\left(\arccos 2\right)$

(j) $\arccos\left(\cos\left(-\dfrac{\pi}{3}\right)\right)$

EXAMPLE 4. *Sketch the graph of* $\arccos(x)$.

- **INVERSE TANGENT:** If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $f(x) = \tan x$ is one-to-one, thus the inverse exists, denoted by $\tan^{-1}(x)$ or $\arctan x$.

	$y = \tan x$	$y = \arctan x = \tan^{-1}(x)$
Domain	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\mathbb{R}$
Range	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$



Cancellation equations:

$$\arctan(\tan x) = x \quad \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

and

$$\tan(\arctan x) = x \quad \text{for all} \quad x.$$

EXAMPLE 5. *Find the exact values of the expression:*

(a) $\arctan 0$ $= 0$, Because $\tan 0 = 0$

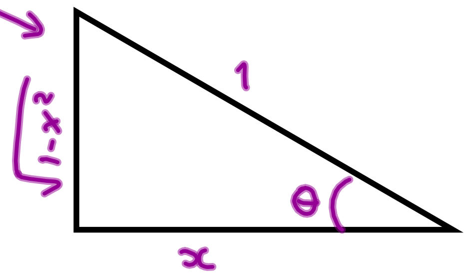
(b) $\arctan(-1)$ $= -\frac{\pi}{4}$ Because $\tan\left(-\frac{\pi}{4}\right) = -1$

(c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ $= -\frac{\pi}{6}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(d) $\tan(\underbrace{\arccos x}_{\theta}) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$

$\cos\theta = x = \frac{x}{1}$



(e) $\arctan\left(\tan\frac{5\pi}{4}\right)$ $= \arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ by Cancellation Rule

$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan \pi/4$

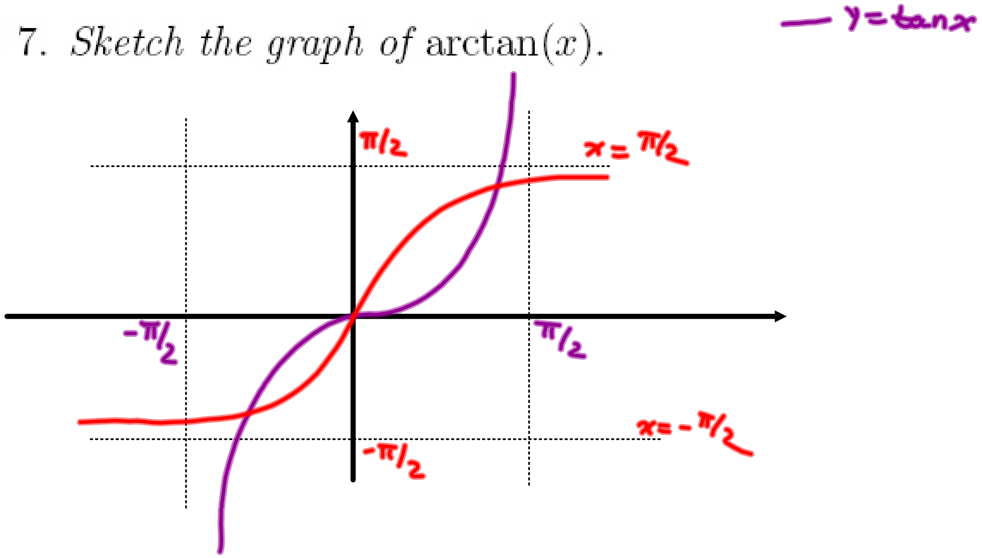
$-\frac{\pi}{2} \le \frac{\pi}{4} \le \pi/2$

EXAMPLE 6. *Find the following limits:*

(a) $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

(b) $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

EXAMPLE 7. *Sketch the graph of* arctan(x).

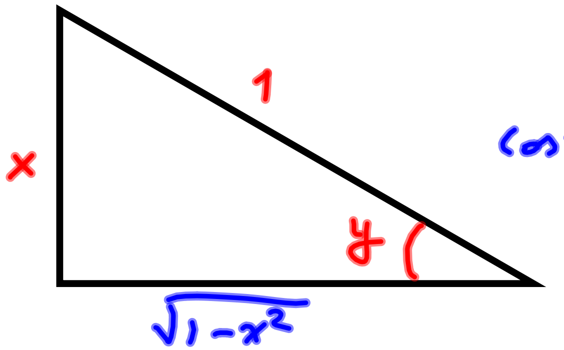


Derivatives of Inverse Trigonometric Functions:

EXAMPLE 8. (a) *Find the derivative of* $f(x) = \arcsin x$.

$y = \arcsin x$

$\boxed{\sin y(x) = x} \Longrightarrow \sin y = \frac{x}{1}$



$\cos y = \sqrt{1-x^2}$

Differentiate implicitely

$(\sin y(x))' = x'$

By chain Rule

$\cos y(x)\, y'(x) = 1$

$y'(x) = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

(b) *Find* $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\arcsin(3x+1)}\right) = -\dfrac{(\arcsin(3x+1))'}{(\arcsin(3x+1))^2}$

Chain Rule

$u = \arcsin(3x+1)$

$$\left(\frac{1}{u(x)}\right)' = -\frac{u'}{u^2}$$

$$= -\frac{\dfrac{3}{\sqrt{1-(3x+1)^2}}}{(\arcsin(3x+1))^2}$$

$$(\arcsin g(x))' = \frac{g'(x)}{\sqrt{1-(g(x))^2}}$$

$$= -\frac{3}{\sqrt{1-(3x+1)^2}\ \arcsin^2(3x+1)}$$

$g(x) = 3x+1$

TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

EXAMPLE 9. *Find the derivative of* $f(x) = \sin^{-1}(\arctan x) = \arcsin(\underline{\arctan x})$

By Chain Rule

$$f'(x) = \frac{1}{\sqrt{1-(\arctan x)^2}} \frac{d}{dx}(\arctan x)$$

$$= \frac{1}{\sqrt{1-(\arctan x)^2}} \cdot \frac{1}{1+x^2}$$

EXAMPLE 10. *Find domain of the following functions:*

(a) $f(x) = \arcsin(4x + 2)$

$u = 4x + 2$

$D(\arcsin u) = [-1, 1] = \{u / -1 \le u \le 1\}$

$-1 \le 4x+2 \le 1$

$-1-2 \le 4x \le 1-2$

$-3 \le 4x \le -1$

$-\frac{3}{4} \le x \le -\frac{1}{4}$

$D(\arcsin(4x+2)) = [-\frac{3}{4}, -\frac{1}{4}]$

(b) $f(x) = \arctan(4x + 2)$

$D(\arctan u) = \mathbb{R}$

$D(\arctan(4x+2)) = \mathbb{R}$