4.8:Indeterminate forms and L'Hospital's Rule

Indeterminate forms: Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}. \tag{1}$$

- If both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then (1) is called an **indeterminate form of type** $\frac{0}{0}$.
- If both $f(x)\to \pm\infty$ and $g(x)\to \pm\infty$ as $x\to a$, then (1) is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

$\frac{f}{g} = \frac{\infty}{\infty}$ $\qquad$ $\frac{f}{g} = \frac{\frac{1}{g}}{\frac{1}{f}} = \frac{0}{0}$

EXAMPLES:

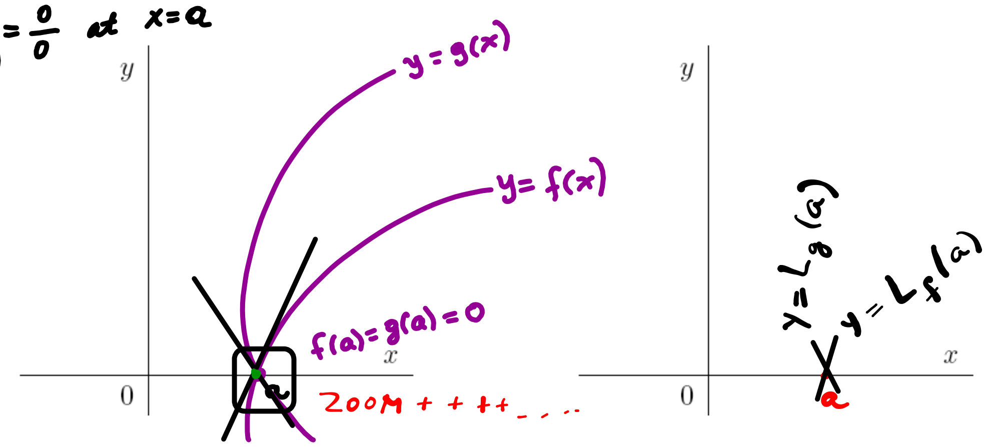
$\lim_{x\to 0}\frac{\sin x}{x} = \frac{0}{0}$, $\quad \lim_{x\to 1}\frac{x-x^2}{x^2-1} = \frac{0}{0}$, $\quad \lim_{x\to 0}\frac{3^x-1}{x^2} = \frac{0}{0}$, $\quad \lim_{x\to\infty}\frac{\ln x}{x^3} = \frac{\infty}{\infty}$, $\quad \lim_{x\to\infty}\frac{x^2+1}{4x^2-1} = \frac{\infty}{\infty}$.

L'HOSPITAL'S RULE: Suppose f and g are differentiable and $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

$\frac{f(x)}{g(x)} = \frac{0}{0}$ at $x=a$



$$f(x) \approx L_f(a) = \cancel{f(a)}^{0} + f'(a)(x-a)$$

$$g(x) \approx L_g(a) = \cancel{g(a)}^{0} + g'(a)(x-a)$$

$$\frac{f(x)}{g(x)} \approx \frac{L_f(a)}{L_g(a)} = \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$$

EXAMPLE 1. *Evaluate each of the following limits:*

(a) $\lim_{x\to 0}\frac{\sin x}{x} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0}\frac{(\sin x)'}{x'} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = \underline{1}$

(b) $\lim_{x\to\infty}\frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty}\frac{(x^2)'}{(e^x)'} = \lim_{x\to\infty}\frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}}$

$= \lim_{x\to\infty}\frac{2}{e^x} = 0$

(c) $\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{-\sin x}{6x}$

$\overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} -\frac{\cos x}{6} = -\frac{\cos 0}{6} = -\frac{1}{6}$

(d) $\lim_{x\to\infty} \frac{(\ln x)^5}{x^4} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{5(\ln x)^4 \cdot \frac{1}{x}}{4x^3} = \frac{5}{4} \lim_{x\to\infty} \frac{(\ln x)^4}{x^4}$

$\overset{\frac{\infty}{\infty}}{\underset{H}{=}} \frac{5}{4} \lim_{x\to\infty} \frac{\cancel{4}(\ln x)^3 \cdot \frac{1}{x}}{\cancel{4}\, x^3} = \frac{5}{4} \lim_{x\to\infty} \frac{(\ln x)^3}{x^4} = \ldots$

$= \ldots = \text{constant} \cdot \underbrace{\lim_{x\to\infty} \frac{1}{x^4}}_{=0} = 0$

Indeterminate form of type $0 \cdot \infty$: $\lim_{x \to a} f(x)g(x) \neq \lim_{x \to a} f'(x)\, g'(x)$

Write the product fg as a quotient to get an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$:

$$g = \frac{1}{\frac{1}{g}}$$

$$fg = 0 \cdot \infty$$

$$= \frac{f}{\frac{1}{g}} = \frac{0}{0}$$

$$= \frac{g}{\frac{1}{f}} = \frac{\infty}{\infty}$$

EXAMPLE 2. *Evaluate each of the following limits:*

(a) $\lim\limits_{x\to 0+} x^2 \ln x \overset{0\cdot\infty}{=}$

$$\frac{x^2}{\frac{1}{\ln x}} = \frac{0}{0}$$

$$\frac{\ln x}{x^{-2}} = \frac{\infty}{\infty} \overset{H}{} \quad \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0^+} -\frac{1}{2x}\cdot x^3$$

$$= \lim_{x\to 0^+} -\frac{x^2}{2} = 0$$

(b) $\lim\limits_{x\to-\infty} xe^x$

$$\frac{e^x}{\frac{1}{x}} = \frac{0}{0}$$

$$\lim_{x\to-\infty} \frac{x}{e^{-x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = -\lim_{x\to-\infty} e^x = 0$$

Indeterminate form of type $\infty - \infty$: $\lim_{x\to a}(f(x) - g(x))$

Try to convert the difference into a quotient to get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

common denominator

rationalizing (multiply by conjugate)

common factor

...

EXAMPLE 3. *Find:* $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) \overset{\infty-\infty}{=}$

$$= \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} \overset{\frac{0}{0}}{\underset{H}{=}}$$

$$= \lim_{x\to 1} \frac{1-0-\frac{1}{x}}{\ln x + \frac{x-1}{x}} \overset{\frac{0}{0}}{=} \lim_{x\to 1} \frac{\frac{x-1}{x}}{\frac{x\ln x + x-1}{x}} = \lim_{x\to 1} \frac{x-1}{x\ln x + x-1}$$

$$\overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 1} \frac{1}{\ln x + x\frac{1}{x} + 1 - 0} = \lim_{x\to 1} \frac{1}{\ln x + 2} = \frac{1}{\ln 1 + 2} = \boxed{\frac{1}{2}}$$

Indeterminate form of type 0^0, ∞^0, 1^∞: $\lim_{x\to a} f(x)^{g(x)}$

Write the function as an exponential $0 \cdot \infty$:

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$$

$a = e^{\ln a}$

$\ln a^b = b \ln a$

It leads to an indeterminate form of type $0 \cdot \infty$.

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}.$$

e^x is continuous function

EXAMPLE 4. *Find the following limits:*

(a) $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = \boxed{1}$

$$x^{\frac{1}{x}} = e^{\ln x^{\frac{1}{x}}} = e^{\frac{1}{x}\ln x}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

(b) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$

$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1 = \boxed{e}$ final answer

$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \overset{\infty\cdot 0}{=}$

$\frac{x}{\frac{1}{\ln\left(1+\frac{1}{x}\right)}}$ NO

$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \underset{\substack{u=\frac{1}{x}\to 0\\ x\to\infty}}{=} \lim_{u\to 0}\frac{\ln(1+u)}{u} \overset{\frac{0}{0}}{\underset{H}{=}}$

$= \lim_{u\to 0}\frac{\frac{1}{1+u}}{1} = 1$

(c) $\lim_{x\to 0+}(\sin x)^{\tan x} \overset{0^0}{=}$

$= e^{\lim_{x\to 0^+}\tan x\,\ln(\sin x)} = e^0 = \boxed{1}$ final answer

$\lim_{x\to 0^+}\tan x\,\ln(\sin x) \overset{0\cdot\infty}{=}$

$\frac{\tan x}{\frac{1}{\ln(\sin x)}}$ NO

$\frac{\ln(\sin x)}{\frac{1}{\tan x}} = \frac{\ln(\sin x)}{\cot x} \overset{\frac{\infty}{\infty}}{=}$

$= \lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x} \underset{H}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} -\frac{\cos x}{\sin x}\sin^2 x$

$= -\lim_{x\to 0^+}\cos x\sin x = -\cos 0\cdot\sin 0 = 0$