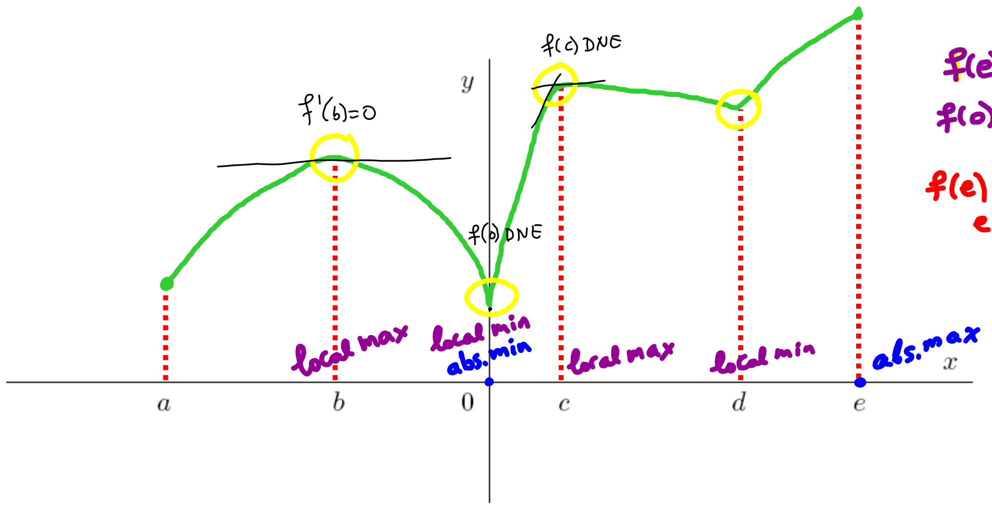
5.2: Maximum and Minimum Values

DEFINITION 1. *Let D be the domain of a function f.*

- *A function f has an* **absolute maximum** *(or* **global maximum***) at $x = c$ if $f(c) \geq f(x)$ for all x in D. In this case, we call $f(c)$ the* **maximum value.**
- *A function f has an* **absolute minimum** *(or* **global minimum***) at $x = c$ if $f(c) \leq f(x)$ for all x in D. In this case, we call $f(c)$ the* **minimum value.**

  The maximum and minimum values of f on D are called the **extreme values** *of f.*

DEFINITION 2. *A function f has a local maximum at $x = c$ if $f(c) \geq f(x)$ when x is near c (i.e. in a neighborhood of c). A function f has a local minimum at $x = c$ if $f(c) \leq f(x)$ when x is near c.*



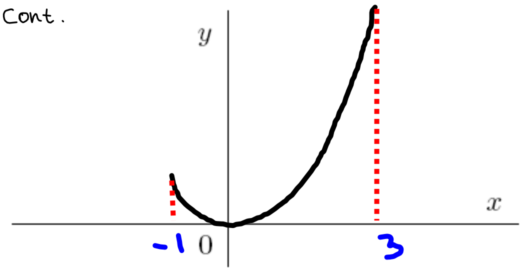
f(e) abs. max. value
f(0) abs. min value

f(e) & f(0) are extreme values

EXAMPLE 3. *Find the absolute and local extrema of f by sketching its graph:*

(a) $f(x) = x^2$, $-1 \leq x \leq 3$ closed interval

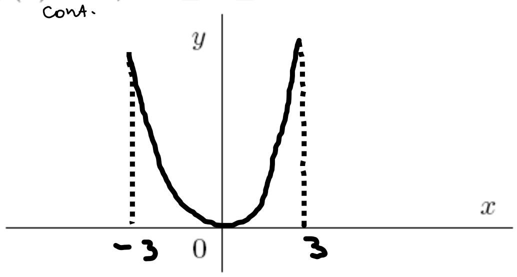
Cont.



	Local	Absolute	Extreme Value
Maximum	NO	$x=3$	9
Minimum	$x=0$	$x=0$	0

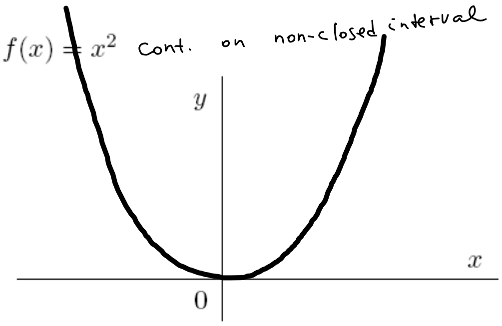
(b) $f(x) = x^2$, $-3 \leq x \leq 3$ closed

Cont.



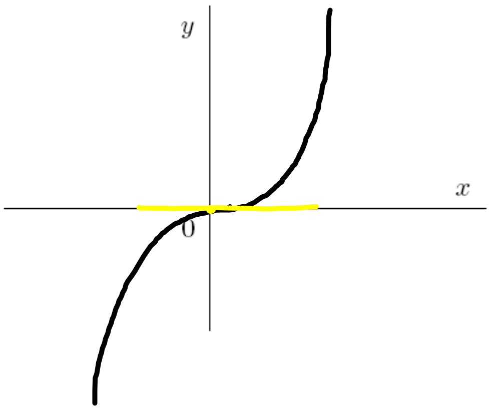
	Local	Absolute	Extreme Value
Maximum	NO	$x=\pm 3$	9
Minimum	$x=0$	$x=0$	0

(c) $f(x) = x^2$ cont. on non-closed interval



	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	x=0	x=0	0

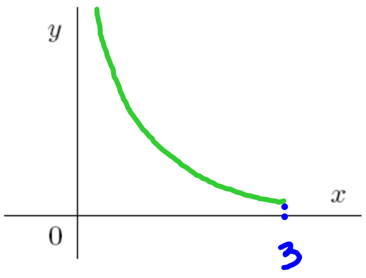
(d) $f(x) = x^3$



	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	NO	NO	NO

cont.

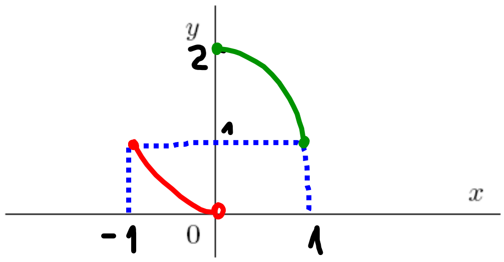
(e) $f(x) = \frac{1}{x}$, $0 \le x \le 3$ non-closed interval



	Local	*Absolute*	*Value*
Maximum	NO	NO	NO
Minimum	NO	$x=3$	$\frac{1}{3}$

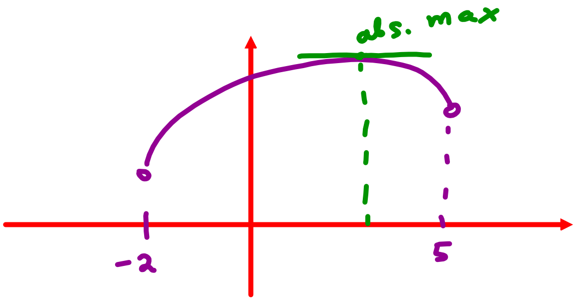
(f) $f(x) = \begin{cases} x^4 & \text{if } -1 \le x < 0 \\ 2 - x^4 & \text{if } 0 \le x \le 1 \end{cases}$ $= [-1, 1]$ closed , but $f(x)$ is not continuous

	Local	*Absolute*	*Value*
Maximum	NO	$x=0$	2
Minimum	NO	NO	NO

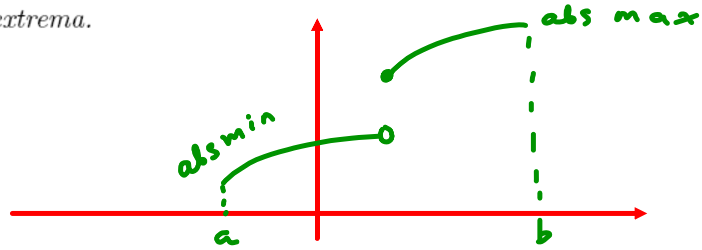


Extreme Value Theorem: If f is a continuous function on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum.

EXAMPLE 4. *Graph an example of a continuous function on a non closed interval that does not attain an an absolute minimum but does attain an absolute maximum.*



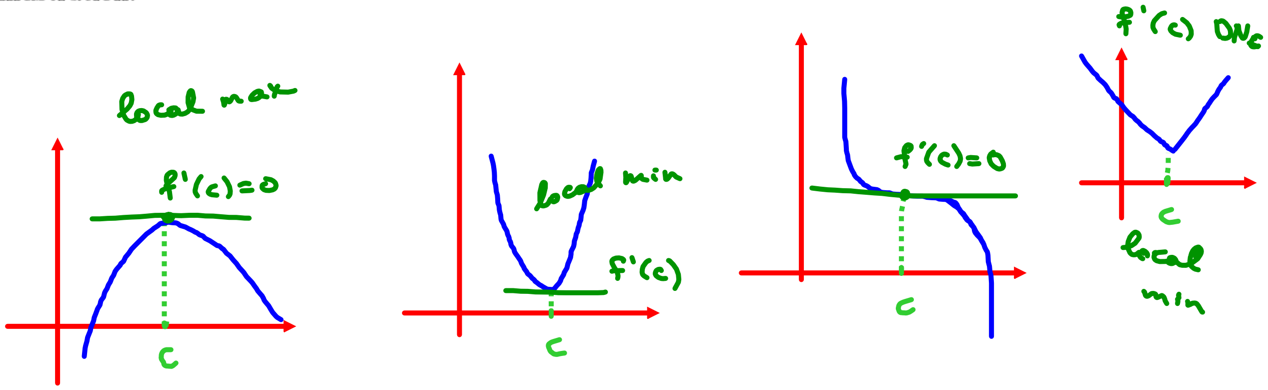
EXAMPLE 5. *Graph an example of a function that is not continuous at a point in the given interval and yet has both absolute extrema.*



point

DEFINITION 6. *A* **critical number** *of $f(x)$ is a number c is in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

Illustration:



EXAMPLE 7. *Find the critical numbers of* $f(x)$*:*

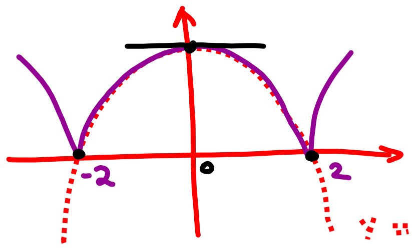
(a) $f(x) = x^3 - 3x^2 + 3x$

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0$

$\boxed{x = 1}$ critical point

Differentiable for all x

(b) $f(x) = |4 - x^2|$



$f'(0) = 0$

$f'(\pm 2)$ DNE

Critical points: $x = 0, \pm 2$

(c) $f(x) = x^{2/5}(5 - x) = 5x^{\frac{2}{5}} - x^{\frac{7}{5}}$

$f'(x)$ DNE if $x = 0$

$f'(x) = 0 \iff 5 \cdot \frac{2}{5} x^{-3/5} - \frac{7}{5} x^{\frac{2}{5}} = 0$

$\frac{10}{x^{3/5}} - 7x^{\frac{2}{5}} = 0 \underset{x \neq 0}{\Rightarrow} \frac{10}{x^{3/5}} = 7x^{2/5}$

$10 = 7x$

$x = \frac{10}{7}$

$\boxed{\text{Critical points: } x = 0, \frac{10}{7}}$

(d) $f(x) = x \ln x$, Note $x > 0$

$f'(x) = \ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1} = \boxed{\frac{1}{e}}$ crit. point

EXAMPLE 8. *Find the absolute extrema for $f(x)$ on the interval I where*

(a) $f(x) = x^3 - 3x^2 + 3x$, $I = [-1, 3]$

Note $f(x)$ is cont. on closed interval

$\Rightarrow$ by EVT f attains both abs. max and abs min

1. Find critical points of f

$$f'(x) = 0 \iff 3x^2 - 6x + 3 = 0$$
$$3(x^2 - 2x + 1) = 0$$
$$3(x-1)^2 = 0$$
$$x = 1$$

2. Find value/s of f at critical point/s which are in I only:

$x = 1$ is in I

$f(1) = 1 - 3 + 3 = \boxed{1}$

3. Find values of f at end-points of I

$$f(-1) = (-1)^3 - 3(-1)^2 + 3(-1) = -1 - 3 - 3 = \boxed{-7}$$

$$f(3) = 3^3 - 3\cdot 3^2 + 3\cdot 3 = \boxed{9}$$

4. Compare the "boxed" values of f:

$\max_I f = f(3) = 9$ abs. max, $\min_I f = f(-1) = -7$ abs. max

(b) $f(x) = \sqrt{3}x^2 + 2\cos x^2$, $I = \left[\frac{\sqrt{\pi}}{2}, \sqrt{\pi}\right]$ closed interval

continuous

① Find critical points

$f'(x) = 2\sqrt{3}\,x - 4x\sin x^2 = 2x(\sqrt{3} - 2\sin x^2) = 0$

$\Rightarrow \underbrace{x = 0}$ OR $\sqrt{3} = 2\sin x^2 \Rightarrow \sin x^2 = \frac{\sqrt{3}}{2}$

$\Rightarrow x^2 = \frac{\pi}{3} + 2\pi n$

$x^2 = \frac{2\pi}{3} + 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots \Rightarrow x = \pm\sqrt{\frac{\pi}{3} + 2\pi n}$

$x = \pm\sqrt{\frac{2\pi}{3} + 2\pi n}$

② Find critical points belonging to I

$x = 0$ is not in I

$I = \left[\frac{\sqrt{\pi}}{2}, \sqrt{\pi}\right] \Rightarrow \frac{\sqrt{\pi}}{2} \le x \le \sqrt{\pi}$

$\frac{\pi}{4} \le x^2 \le \pi$

The only critical points in I are

$x = \sqrt{\frac{\pi}{3}}, \sqrt{2\pi/3}$

③ Find values of f at critical points in I:

$$f\left(\sqrt{\frac{\pi}{3}}\right) = \sqrt{3}\cdot\frac{\pi}{3} + 2\cos\frac{\pi}{3} = \frac{\pi\sqrt{3}}{3} + 1 \approx \boxed{2.81}$$

$$f\left(\sqrt{\frac{2\pi}{3}}\right) = \frac{2\sqrt{3}\pi}{3} + 2\cos\frac{2\pi}{3} = \frac{2\sqrt{3}\pi}{3} - 1 \approx \boxed{2.63}$$

④ Find values of f at end points

$$f\left(\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{3}\pi}{4} + 2\cos\frac{\pi}{4} = \frac{\sqrt{3}\pi}{4} + \sqrt{2} \approx \boxed{2.77}$$

$$f(\sqrt{\pi}) = \sqrt{3}\pi + 2\cos\pi = \sqrt{3}\pi - 1 \approx \boxed{4.44}$$

⑤ Conclusion

$$\max_I f = f(\sqrt{\pi}) \approx 4.44$$

$$\min_I f = f\left(\sqrt{\frac{2\pi}{3}}\right) \approx 2.63$$