5.3: Derivatives and Shapes of Curves

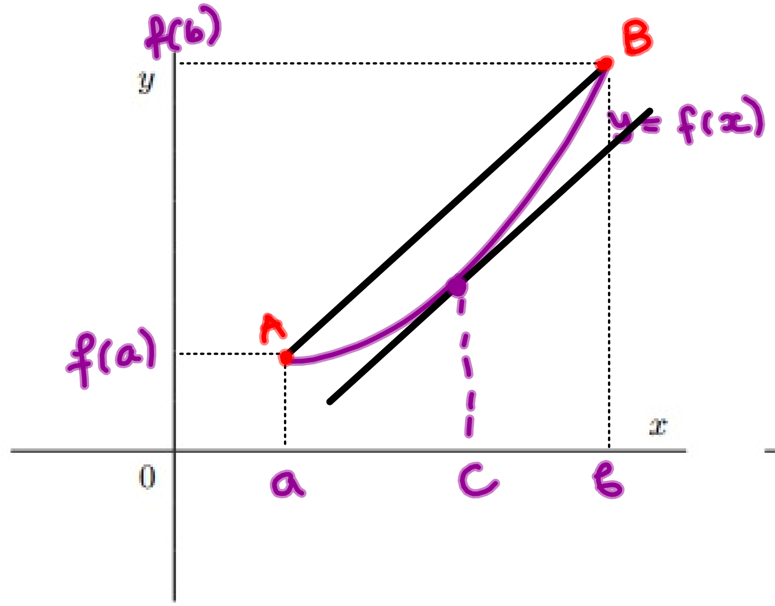
Mean Value Theorem: Suppose a function f is continuous on the (closed) interval $[a, b]$ and differentiable on the (open) interval (a, b). Then there is a number c such that $a < c < b$ and

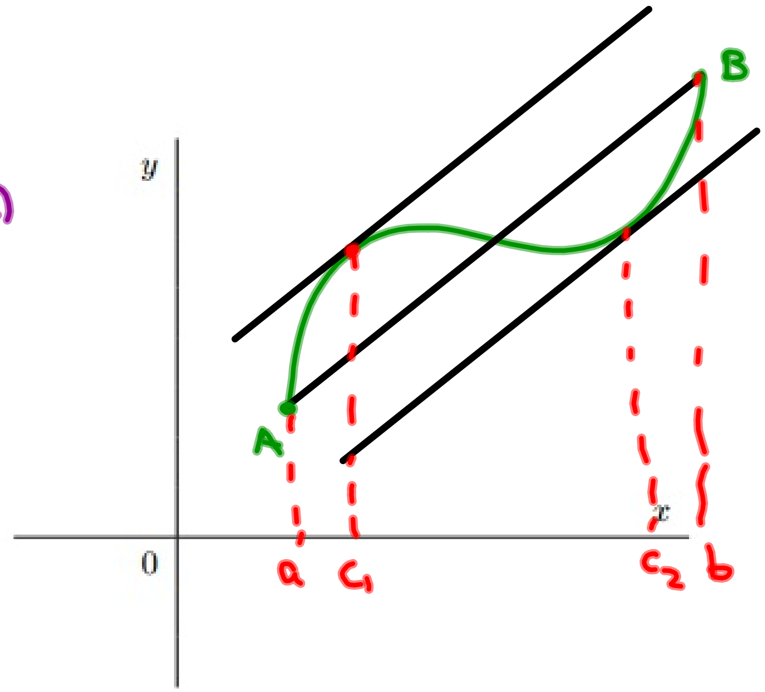
slope of tangent at $x=c$ $= f'(c) = \dfrac{f(b) - f(a)}{b - a}$ = slope of the secant AB

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Illustration: $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$





EXAMPLE 1. *Find a number c that satisfies the conclusion of the Mean Value Theorem on the interval $[0,2]$ when $f(x) = x^3 + x - 1$.*

MVT

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

In our case:

$$f'(x) = 3x^2 + 1 \Rightarrow f'(c) = 3c^2 + 1$$

$$a = 0, \quad b = 2$$

$$f(0) = -1, \quad f(2) = 8 + 2 - 1 = 9$$

$$\Rightarrow 3c^2 + 1 = \frac{9-(-1)}{2-0}$$

$$3c^2 + 1 = 5$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm \frac{2}{\sqrt{3}}$$

$$c = \pm \frac{2\sqrt{3}}{3}$$

Final answer is $c = \frac{2\sqrt{3}}{3}$ because $0 < c < 2$.

EXAMPLE 2. *Suppose* $\boxed{1 \le f'(x) \le 4}$ $\boxed{\textit{for all } x}$ *in the* [2, 5]. *Show that* $3 \le f(5) - f(2) \le 12$.

MVT:

$$f(b) - f(a) = f'(c)(b-a)$$

$$f(5) - f(2) = f'(c)(5-2)$$

$\Rightarrow$

In particular,

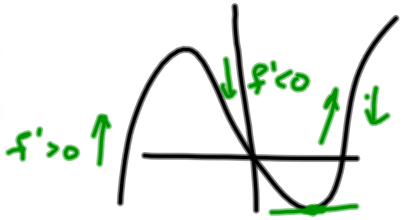
$1 \le f'(c) \le 4$ because $2 < c < 5$

$$\Rightarrow \quad 3 \cdot 1 \le f(5) - f(2) = 3f'(c) \le 3 \cdot 4$$

$$\boxed{3 \le f(5) - f(2) \le 12}$$

Test for increasing/decreasing

- If $f'(x) > 0$ on an interval , then f is increasing on that interval.
- If $f'(x) < 0$ on an interval , then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval , then f is constant on that interval.



EXAMPLE 3. *Determine all intervals where the following function*

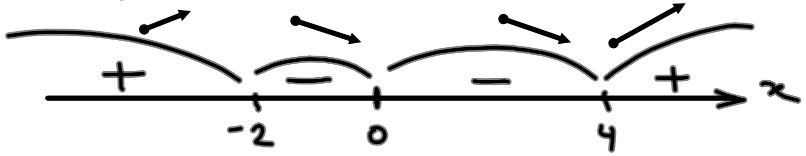
$$f(x) = x^5 - \frac{5}{2}x^4 - \frac{40}{3}x^3 - 12$$

is increasing or decreasing.

$f'(x) = 5x^4 - 10x^3 - 40x^2 = 5x^2(x^2 - 2x - 8)$

$= 5x^2(x+2)(x-4) = 0$

$x=0$, $x=-2$, $x=4$ are critical points



$f(x)$ is increasing on $(-\infty, -2) \cup (4, +\infty)$

$f(x)$ is decreasing on $(-2, 4)$

First Derivative Test: Suppose that $x = c$ is a critical point of a continuous function f.

- If $f'(x)$ changes from negative to positive at $x = c$, then f has a local minimum at c.
- If $f'(x)$ changes from positive to negative at $x = c$, then f has a local maximum at c.
- If $f'(x)$ does not change sign at $x = c$, then f has no local maximum or minimum at c.

REMARK 4. The first derivative test only classifies critical points as local extrema and not as absolute extrema.

EXAMPLE 5. *For function from Example 3 identify all local extrema.*

$x = 4$ local min

$x = -2$ local max

EXAMPLE 6. *Find all intervals of increase and decrease of f and identify all local extrema.*

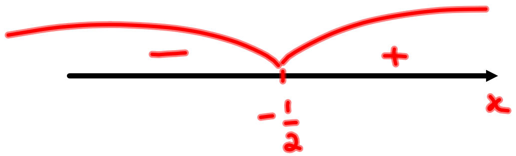
(a) $f(x) = xe^{2x}$

$$f'(x) = e^{2x} + 2xe^{2x} = \underbrace{e^{2x}}_{\neq 0}(1+2x) = 0$$

$$\Downarrow$$

$1+2x=0$

$x=-\frac{1}{2}$ is critical point



$$\left.\begin{array}{l} f(x) \nearrow \text{ on } \left(-\frac{1}{2}, \infty\right) \\ f(x) \searrow \text{ on } \left(-\infty, -\frac{1}{2}\right) \end{array}\right\} \Rightarrow x=-\frac{1}{2} \text{ is local min}$$

(b) $f(x) = x\sqrt[3]{x^2 - \frac{5}{3}} = x\left(x^2 - \frac{5}{3}\right)^{\frac{1}{3}}$

$$f'(x) = \left(x^2 - \frac{5}{3}\right)^{\frac{1}{3}} + x\,\frac{1}{3}\left(x^2 - \frac{5}{3}\right)^{\frac{1}{3}-1} \cdot 2x$$

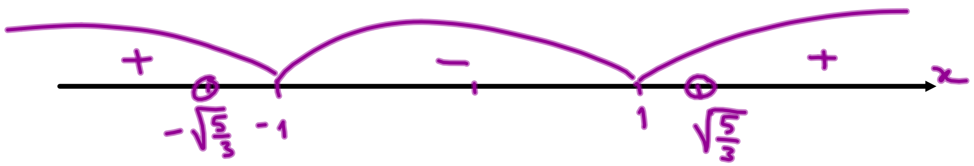
$$= \left(x^2 - \frac{5}{3}\right)^{\frac{1}{3}} + \frac{2x^2}{3}\left(x^2 - \frac{5}{3}\right)^{-\frac{2}{3}}$$

$$= \left(x^2 - \frac{5}{3}\right)^{\frac{1}{3}} + \frac{2x^2}{3\left(x^2 - \frac{5}{3}\right)^{2/3}}$$

$$= \frac{3\left(x^2 - \frac{5}{3}\right) + 2x^2}{3\left(x^2 - \frac{5}{3}\right)^{2/3}} = \frac{3x^2 - 5 + 2x^2}{3\left(x^2 - \frac{5}{3}\right)^{2/3}}$$

$$= \frac{5(x^2-1)}{3\left(x^2 - \frac{5}{3}\right)^{2/3}} = \frac{5(x-1)(x+1)}{3\left(\left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right)\right)^{2/3}}$$

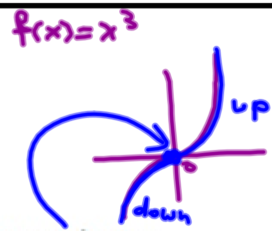
Critical points: $x = 1, x = -1, x = \sqrt{\frac{5}{3}}, x = -\sqrt{\frac{5}{3}}$



$f(x) \nearrow$ on $(-\infty, -1) \cup (1, +\infty)$
$f(x) \searrow$ on $(-1, 1)$

$\Rightarrow$ $x = -1$ is local max
$x = 1$ is local min

Recall here the **Second derivative test for concavity.** (see Section 5.1):



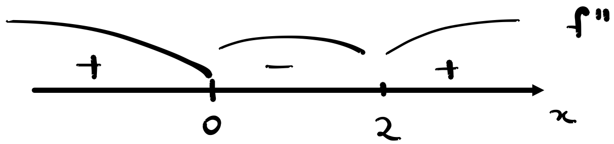
- If $f''(x) > 0$ for all x on an interval, then f is concave up on that interval.
- If $f''(x) < 0$ for all x on an interval, then f is concave down on that interval.

In addition, if f changes concavity at $x = a$, and $x = a$ is in the domain of f, then $x = a$ is an inflection point of f.

EXAMPLE 7. *Find intervals of concavity and inflection points of f, if $f'(x) = 4x^3 - 12x^2$.*

$$f''(x) = f'(f'(x)) = (4x^3 - 12x^2)' = 12x^2 - 24x$$

$$= 12x(x-2)$$



$f(x)$ is concave up on $(-\infty, 0) \cup (2, +\infty)$

$f(x)$ is concave down on $(0, 2)$

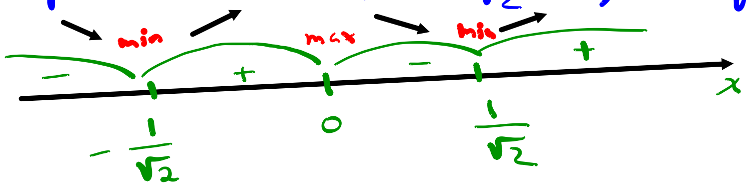
$x=0$ and $x=2$ are inflection points.

EXAMPLE 8. *Sketch the graph of $f(x) = x^4 - x^2$ by locating intervals of increase/decrease, local extrema, concavity and inflection points.*

Locate intervals of $\nearrow$ $\searrow$ and local extrema

$$f'(x) = 4x^3 - 2x = 2x(2x^2-1) = 2x(\sqrt{2}\,x-1)(\sqrt{2}\,x+1)$$

Critical points $x=0$, $x=\frac{1}{\sqrt{2}}$, $x=-\frac{1}{\sqrt{2}}$

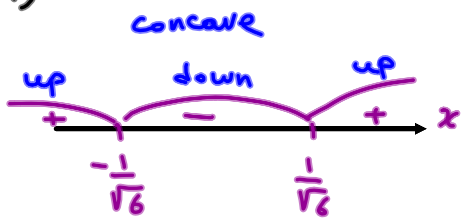


Locate intervals of concavity and inflection points

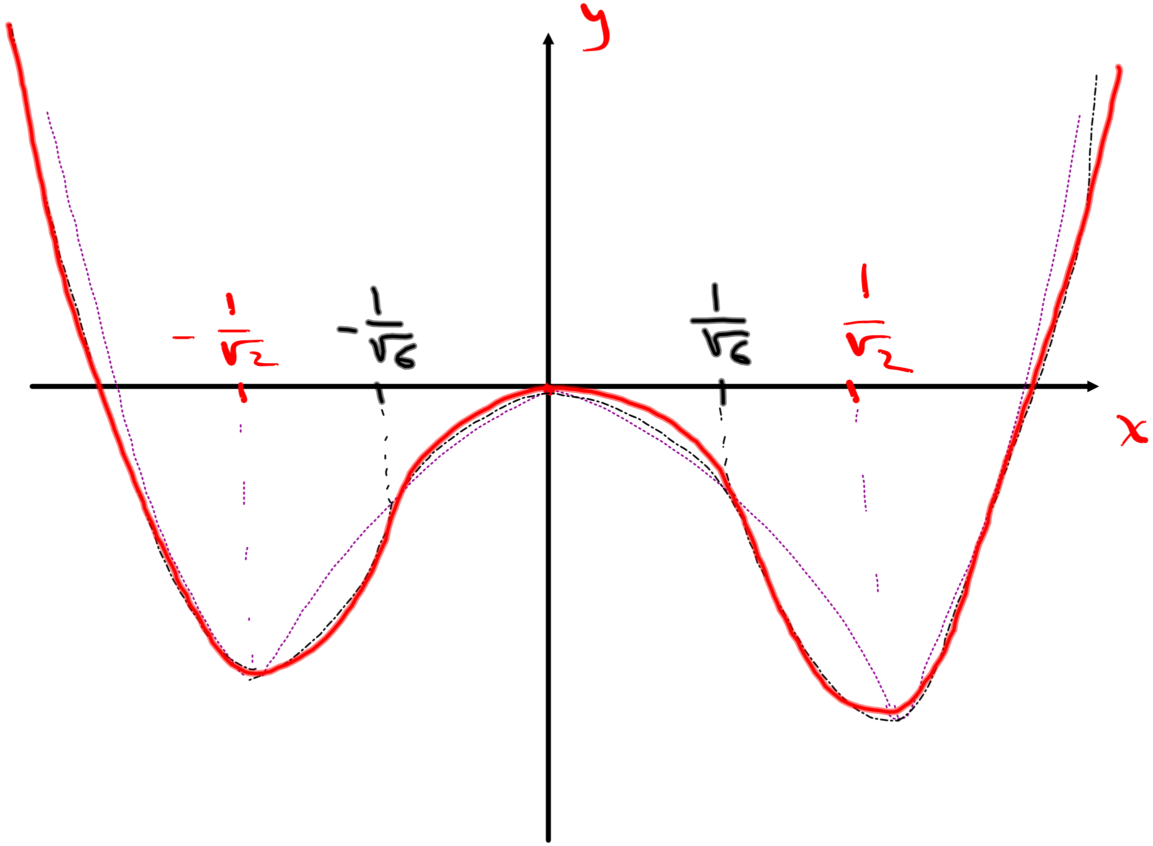
$$f''(x) = (4x^3-2x)' = 12x^2-2 = 2(6x^2-1)$$

$$= 2(\sqrt{6}\,x-1)(\sqrt{6}\,x+1)$$

$f''(x)=0$ if $x=\pm\frac{1}{\sqrt{6}}$



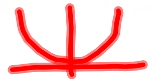
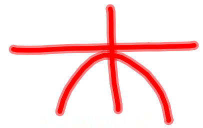
Note $f(0)=0$



Note $f(-x) = f(x)$, i.e. $f(x)$ is even

$\Rightarrow$ the graph is symmetric w.r.t. the y-axis.

Second derivative test for local extrema: Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c. 
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c. 

REMARK 9. If $f'(c) = 0$ and $f''(c) = 0$ or does not exist, then the test fails. In the case $f''(c)$ does not exist we use the first derivative test to find the local extrema.

EXAMPLE 10. *Find the local extrema for* $f(x) = 1 - 3x + 5x^2 - x^3$.

$$f'(x) = -3 + 10x - 3x^2 = -(3x^2 - 10x + 3)$$

$$= -3\left(x - \tfrac{1}{3}\right)(x-3) = 0$$

$x = \frac{1}{3}$ $\quad$ $x = 3$ $\quad$ Critical points

$$f''(x) = 10 - 6x$$

$f''\left(\frac{1}{3}\right) = 10 - 6 \cdot \frac{1}{3} > 0 \Rightarrow x = \frac{1}{3}$ is local min

$f''(3) = 10 - 6 \cdot 3 < 0 \Rightarrow x = 3$ is local max