5.5: Applied Maximum and Minimum Problems

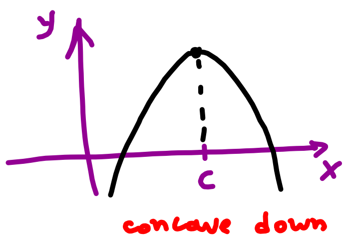
OPTIMIZATION PROBLEMS

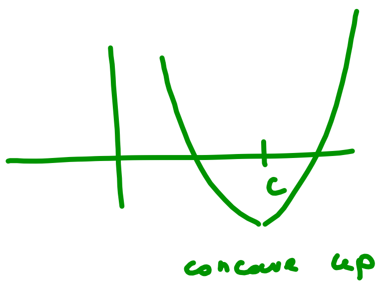
First derivative test for absolute extrema: Suppose that c is a critical number of a continuous function f defined on an interval.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Alternatively,

- *If $f''(x) < 0$ for all x (so f is always concave downward) then the local maximum at c must be an absolute maximum.*
- *If $f''(x) > 0$ for all x (so f is always concave upward) then the local minimum at c must be an absolute minimum.*





EXAMPLE 1. *When a producer sells x items per week, he makes a profit of*

$$p(x) = 15x - 0.001x^2 - 2000.$$

How many items does he need to sell to get the maximum profit?

$p(x) \to \max$ when $x \geq 0$

Find critical points when $x \geq 0$:

$p'(x) = 15 - 0.002x = 0$

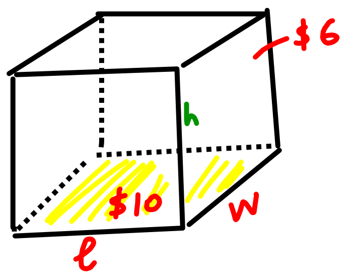
$15 = 0.002x \Rightarrow x = \frac{15}{0.002} = \frac{15000}{2} = 7500$

$\boxed{x = 7500}$ critical point

$p''(x) = -0.002 < 0 \Rightarrow$ abs. max at $x = 7500$ ✓

The producer needs to sell 7500 items to get the maximum profit.

EXAMPLE 2. *A rectangular storage container with an open top is to have a volume of* 10m^3. *The length of its base is twice the width. Material for the base costs* \$10 *per squire meter. Material for the sides costs* \$6 *per square meter. Find the cost of materials for the cheapest such container.*



$$V = 10 = \ell w h = 2w^2 h$$

$$\ell = 2w$$

$$\Downarrow$$

$$h = \frac{10}{2w^2}$$

$$h = \frac{5}{w^2}$$

$$C = 10\ell w + 2\cdot 6(\ell h + hw)$$

$$= 10\ell w + 12h(\ell + w)$$

$$C(w) = 10\, 2w\cdot w + 12\cdot\frac{5}{w^2}\overbrace{(2w+w)}^{3w}$$

$$C(w) = 20w^2 + \frac{180\cancel{w}}{w^{\cancel{2}}}$$

$$C(w) = 20\left(w^2 + \frac{9}{w}\right) \to \text{min} \quad \text{when } 0<w$$

Critical points: $C'(w) = 20\left(2w - \frac{9}{w^2}\right) = 0$

Note $w \neq 0$ $\qquad 2w = \frac{9}{w^2} \Rightarrow w^3 = \frac{9}{2} \Rightarrow w = \boxed{\sqrt[3]{\frac{9}{2}}}$ Crit. point

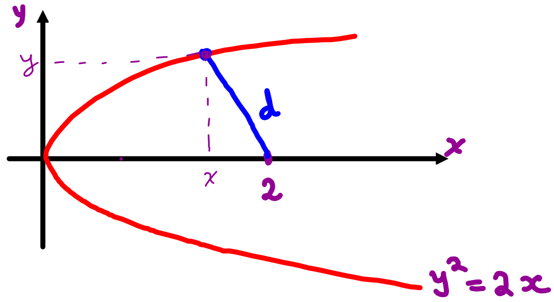
$$C''(w) = 20\left(2 + \frac{9\cdot 2}{w^3}\right) > 0 \quad \text{when } w > 0$$

$\Rightarrow$ at $w = \sqrt[3]{\frac{9}{2}}$ the function $C(w)$ has abs. min.

Cost of the cheapest box:

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{81}{4}} + 9\sqrt[3]{\frac{2}{9}}\right) = \$\ldots$$

EXAMPLE 3. *Find the shortest distance from the parabola* $y^2 = 2x$ *to the point* $(2, 0)$.



$$d\big((x,y),(2,0)\big) = \sqrt{(x-2)^2 + (y-0)^2}$$

$$d = \sqrt{(x-2)^2 + y^2} \quad \text{, where } y^2 = 2x$$

$$d = \sqrt{(x-2)^2 + 2x} = \sqrt{x^2 - 4x + 4 + 2x}$$

$$d(x) = \sqrt{x^2 - 2x + 4} \longrightarrow \text{min when } x \geq 0.$$

$$\min_{x \geq 0} d(x) = \sqrt{\min_{x \geq 0} d^2(x)} \qquad (\text{Note } d(x) \geq 0 \text{ for all } x).$$

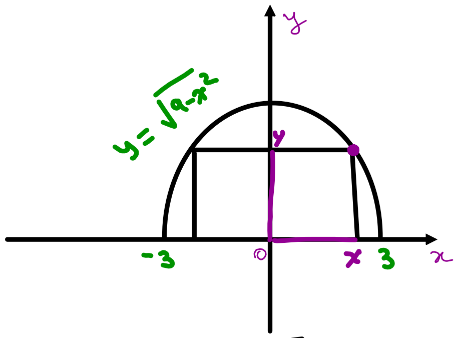
$$f(x) = d^2(x) \quad \Rightarrow \quad f(x) = x^2 - 2x + 4$$

Critical points: $f'(x) = 2x - 2 = 2(x-1) = 0 \Rightarrow \boxed{x = 1}$

$f''(x) = 2 > 0 \quad \Rightarrow$ at $x = 1$ there is abs. min

Shortest distance: $d(1) = \sqrt{1^2 - 2 \cdot 1 + 4} = \sqrt{3}$

EXAMPLE 4. *A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$. What length and width should the rectangle have so that its area is a maximum?* (Equivalently, find the dimensions of the largest rectangle that can be inscribed in the semi-disk with radius 3.)



$$A = 2xy = 2x\sqrt{9-x^2}$$

$$A(x) = 2x\sqrt{9-x^2} \longrightarrow \max \quad \text{when } 0<x<3$$

Critical points

$$A'(x) = 2\left[\sqrt{9-x^2} + \frac{x\cdot(-\cancel{2}x)}{\cancel{2}\sqrt{9-x^2}}\right]$$

$$= 2\left[\frac{9-x^2-x^2}{\sqrt{9-x^2}}\right] = \frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$$

Note $x \neq \pm 3$.

$$9-2x^2=0$$
$$2x^2=9$$
$$x^2=\frac{9}{2}$$
$$x = \pm\frac{3}{\sqrt{2}}$$

Only one critical point $x=\frac{3}{\sqrt{2}}$ on $(0,3)$.

$$A''(x) = 2\left(\frac{9-2x^2}{\sqrt{9-x^2}}\right)'$$

$$A''(x) = \frac{2\left(-4x\sqrt{9-x^2} - (9-2x^2)\frac{-x}{\sqrt{9-x^2}}\right)}{9-x^2}$$

$$A''(x) = \frac{2}{9-x^2}\left(\frac{-4x(9-x^2) + (9-2x^2)x}{\sqrt{9-x^2}}\right)$$

$$= \frac{2x}{(9-x^2)^{3/2}}\left[-36+4x^2+9-2x^2\right] = \underbrace{\frac{2x}{(9-x^2)^{3/2}}}_{>0 \text{ when } 0<x<3}\underbrace{(2x^2-27)}_{<0}$$

$$0<x<3 \Rightarrow 0<x^2<9$$
$$0<2x^2<18$$
$$-27<2x^2-27<18-27$$
$$\Downarrow$$
$$2x^2-27<0$$

$A''(x)<0$ on $(0,3)$

$\Downarrow$

at $x=\frac{3}{\sqrt{2}}$ $A(x)$ has abs. max

Dimensions of the rectangle with max area:

$$\text{length} = 2x = 2\cdot\frac{3}{\sqrt{2}} = \boxed{3\sqrt{2}}$$

$$\text{width} = y = \sqrt{9-x^2} = \sqrt{9-\frac{9}{2}} = \boxed{\frac{3}{\sqrt{2}}}$$