Section 5.7: Antiderivatives

DEFINITION 1. *A function F is called an* **antiderivative** *of f on an interval I if $F'(x) = f(x)$ for all x in I.*

EXAMPLE 2. **(a)** *Is the function $F(x) = x\ln(x) - x + \sin x$ is an antiderivative of* *$f(x) = \ln(x) + \cos x$?*

$$F'(x) = \ln x + x\frac{1}{x} - 1 + \cos x = \ln x + \cos x = f(x)$$

YES

(b) *Is the function $F_1(x) = x\ln(x) - x + \sin x + 10$ is an antiderivative of $f(x) = \ln(x) + \cos x$?*

$$F_1'(x) = (F(x) + 10)' = F'(x) + (10)' = f(x) + 0 = f(x) \quad \text{YES}$$

(c) *What is the most general antiderivative of $f(x) = \underline{\ln(x)} + \underline{\cos x}$?*

$$F(x) + C = \underline{x\ln x - x} + \underline{\sin x} + C$$

where C is an arbitrary constant.

THEOREM 3. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.*

EXAMPLE 4. *Find the most general antiderivative of $f = 2x$.*

Find $F(x)$ such that $F'(x) = 2x$

$$\boxed{F(x) = x^2 + C}$$

Properties of antiderivative

$$(F + G)' = F' + G'$$

α is real $\quad (\alpha F)' = \alpha F'$

Note $\quad (FG)' \neq F'G'$

$$\left(\frac{F}{G}\right)' \neq \frac{F'}{G'}$$

Table of Antidifferentiation Formulas

$F'(x) = f(x)$

Function, $f(x)$	Particular antiderivative $F(x)$	Most general antiderivative $F(x) + C$
$k \quad (k \in \mathbb{R})$	kx	$kx + C$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\cos x$	$\sin x$	
$\sin x$	$-\cos x$	
$\sec^2 x$	$\tan x$	$+ C$
$\csc^2 x$	$-\cot x$	
$\sec x \tan x$	$\sec x$	
$\csc x \cot x$	$\csc x$	
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	
$\dfrac{1}{1+x^2}$	$\arctan x$	

EXAMPLE 5. *Find the most general antiderivative of f whe*

(a) $f(x) = 5\sin x + 7x^6 - \sqrt[8]{x^7} + 15$ $\quad x^{7/8}$

$$F(x) = 5(-\cos x) + 7\frac{x^{6+1}}{6+1} - \frac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + 15x + C$$

$$F(x) = -5\cos x + x^7 - \frac{8}{15}x^{15/8} + 15x + C$$

(b) $f(x) = \dfrac{3x + 8 - x^2}{x^3} = \dfrac{3x}{x^3} + \dfrac{8}{x^3} - \dfrac{x^2}{x^3} = 3x^{-2} + 8x^{-3} - \dfrac{1}{x}$

$$F(x) = 3\frac{x^{-2+1}}{-2+1} + 8\frac{x^{-3+1}}{-3+1} - \ln|x| + C$$

$$F(x) = -\frac{3}{x} - \frac{4}{x^2} - \ln|x| + C$$

(c) $f(x) = e^x + (1 - x^2)^{-1/2} = e^x + \dfrac{1}{\sqrt{1-x^2}}$

$$F(x) = e^x + \arcsin(x) + C$$

EXAMPLE 6. *Find $f(x)$ given that $f'(x) = 4 - 3(1+x^2)^{-1}$, $f(1) = 0$.*

$f(x)$ is antiderivative of $f'(x)$

$f(x) = 4x - 3\arctan x + C$

$0 = f(1) = 4 - 3\arctan 1 + C$

$0 = 4 - \frac{3\pi}{4} + C$

$C = \frac{3\pi}{4} - 4$

$$\boxed{f(x) = 4x - 3\arctan x + \frac{3\pi}{4} - 4}$$

EXAMPLE 7. *Find* $f(x)$ *given that* $f''(x) = 3e^x + 5sinx$, $f(0) = 1$, $f'(0) = 2$.

$f'(x)$ is an antiderivative of $f''(x) = (f'(x))'$

$f'(x) = 3e^x - 5\cos x + C$

$f(x)$ is an antiderivative of $f'(x)$

$f(x) = 3e^x - 5\sin x + Cx + C_1$

Use the initial conditions $f(0) = 1$, $f'(0) = 2$ to determine C and C_1:

$$\left.\begin{array}{l} 1 = f(0) = 3e^0 - 5\sin 0 + C\cdot 0 + C_1 \\ 2 = f'(0) = 3e^0 - 5\cos 0 + C \end{array}\right\} \Rightarrow \begin{array}{l} 1 = 3 + C_1 \\ 2 = 3 - 5 + C \end{array} \Rightarrow$$

$$\Rightarrow \begin{array}{l} C_1 = -2 \\ C = 4 \end{array}$$

$$\boxed{f(x) = 3e^x - 5\sin x + 4x - 2}$$

EXAMPLE 9. *Show that for motion in a straight line with a constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is*

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0.$$

Given

$a(t) = a$

$v(0) = v_0$

$s(0) = s_0$

We know

$s'(t) = v(t)$

$s''(t) = v'(t) = a(t)$

We have

$s''(t) = a$

$v(t) = s'(t) = at + C \quad (1)$

$s(t) = \frac{at^2}{2} + Ct + C_1 \quad (2)$

Use these initial conditions to find C and C_1

$v_0 = v(0) \overset{(1)}{=} a \cdot 0 + C \Rightarrow \boxed{C = v_0}$

$s_0 = s(0) \overset{(2)}{=} \frac{a \cdot 0^2}{2} + C \cdot 0 + C_1 \Rightarrow \boxed{C_1 = s_0}$

$$\boxed{s(t) = \frac{at^2}{2} + v_0 t + s_0}$$