6.1: Sigma notation

DEFINITION 1. *If $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ are real numbers and m and n are integers such that $m \le n$, then*

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n = \sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_j$$

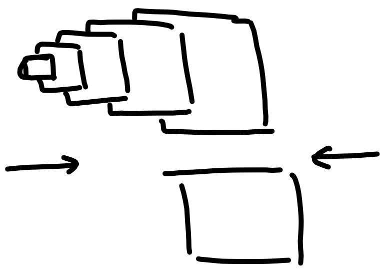
EXAMPLE 2. *Compute these summations:*

(a) $$\sum_{i=5}^{8} \underbrace{(\sin i - \sin(i+1))}_{a_i} = \underbrace{\sin 5 - \cancel{\sin 6}}_{a_5} + \underbrace{\cancel{\sin 6} - \cancel{\sin 7}}_{a_6} + \underbrace{\cancel{\sin 7} - \cancel{\sin 8}}_{a_7} + \underbrace{\cancel{\sin 8} - \sin 9}_{a_8}$$

$i = 5, 6, 7, 8$

$$= \sin 5 - \sin 9$$

Telescoping sum



Question:

$$\sum_{i=5}^{8} (\cos i - \cos(i+1)) = \cos 5 - \cos 9$$

(b) $$\sum_{i=1}^{10} 3 = \underbrace{3 + 3 + \ldots + 3}_{10 \text{ times}} = 3 \cdot 10 = 30$$

EXAMPLE 3. *Write the sum in sigma notation:*

$$-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} = -\underset{k=1}{\frac{1}{1^2}} + \underset{k=2}{\frac{1}{2^2}} - \underset{k=3}{\frac{1}{3^2}} + \underset{k=4}{\frac{1}{4^2}} - \underset{k=5}{\frac{1}{5^2}}$$

$$= \sum_{k=1}^{5} \frac{(-1)^k}{k^2}$$

Note:

$$(-1)^k = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}$$

$$(-1)^{k+1} = (-1)^{k-1}$$

$$(-1)^{k+2} = (-1)^k$$

THEOREM 4. *If c is any constant then*

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Note that in general

$$\sum_{i=m}^{n} a_i b_i \neq \left(\sum_{i=m}^{n} a_i\right) \cdot \left(\sum_{i=m}^{n} b_i\right).$$

$$\sum_{i=m}^{n} \frac{a_i}{b_i} \neq \frac{\sum a_i}{\sum b_i}$$

$$\sum_{i=m}^{n} 1 = n - m + 1$$

$$\sum_{i=4}^{6} 1 = 1 + 1 + 1 = 3 = 2 + 1$$

EXAMPLE 5. *If* $\sum_{i=1}^{25} f(i) = 15,\ f(25) = 7$ *and* $\sum_{i=1}^{24} g(i) = 25$ *find* $\sum_{i=1}^{24} (2f(i) - g(i))$

$$\sum_{i=1}^{24} (2f(i) - g(i)) = 2\sum_{i=1}^{24} f(i) - \underbrace{\sum_{i=1}^{24} g(i)}_{25} = 2\cdot 8 - 25 = 16 - 25 = \boxed{-9}$$

$$15 = \sum_{i=1}^{25} f(i) = \underbrace{f(1) + f(2) + \dots + f(24)}_{\sum_{i=1}^{24} f(i)} + \underbrace{f(25)}_{7}$$

$$15 = \sum_{i=1}^{24} f(i) + 7 \Rightarrow \sum_{i=1}^{24} f(i) = 15 - 7 = 8$$

- $\sum_{i=1}^{n} 1 = n$ Note $\sum_{i=m}^{n} 1 = n-m+1$

- $\sum_{i=1}^{n} c = nc$, *where c is a constant.* $\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = cn$

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Principle of Mathematical Induction

Base of Induction: The statement P(1) is true.

Step: Assume that P(n) is true and derive that P(n+1) is true.

Then P(n) is true for all n.

Here n is positive integer.

Example Use induction to prove

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof

$P(n): \quad 1+2+3+\dots+n = \frac{n(n+1)}{2}$

Base $P(1): \quad 1 = \frac{1(1+1)}{2} \Rightarrow 1 = \frac{1\cdot 2}{2} \Rightarrow 1=1$ (true)

Assume that P(n) is true.

In other words, assume that

$P(n): \quad 1+2+3+\dots+n = \frac{n(n+1)}{2}$ is true.

We will derive that then P(n+1) is true.

$P(n+1) \stackrel{?}{:} \quad \underbrace{1+2+3+\dots+n+(n+1)}_{LHS} = \underbrace{\frac{(n+1)(n+2)}{2}}_{RHS}$ Verify

$$LHS: \underbrace{\boxed{1+2+3+\dots+n}}_{P(n)} +(n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2} = RHS.$$

Thus P(n+1) is true (independently of n). Hence, P(n) is true for all positive integers n by Principle of Math. Induction.

Ex Prove by induction the following formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof Denote

$$P(n): \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Base of induction:

$$P(1): \quad 1^2 = \frac{1\cdot(1+1)\,(2\cdot 1+1)}{6} \Rightarrow 1 = \frac{2\cdot 3}{6} \text{ (TRUE)}$$

Assume that $P(n): \; 1^2+2^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true

Show that $P(n+1): \; 1^2+2^2+\dots+n^2+(n+1)^2 = \dfrac{(n+1)(n+1+1)(2(n+1)+1)}{6}$

OR

$$P(n+1): \; \underbrace{1^2+2^2+\dots+n^2+(n+1)^2}_{LHS} = \underbrace{\frac{(n+1)(n+2)(2n+3)}{6}}_{RHS}$$

is true.

$$LHS = \underbrace{\boxed{1^2+2^2+\dots+n^2}}_{P(n)} + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\big(n(2n+1) + 6(n+1)\big)}{6}$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6} = \frac{(n+1)(2n^2+7n+6)}{6}$$

Note
$(n+2)(2n+3) \overset{\text{FOIL}}{=}$
$2n^2+6+4n+3n$
$=2n^2+7n+6$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = RHS$$

Thus $P(n) \Longrightarrow P(n+1)$ and by Induction $P(n)$ is true for all n.

EXAMPLE 7. *Compute these sums:*

(a) $\sum_{i=4}^{27} 20 = 20 \sum_{i=4}^{27} 1 = 20(27-4+1) = 20 \cdot 24 = 480$

(b) $\sum_{i=1}^{n} i(i+4) = \sum_{i=1}^{n} (i^2 + 4i) = \sum_{i=1}^{n} i^2 + 4 \sum_{i=1}^{n} i$ by theorem 6

$$= \frac{n(n+1)(2n+1)}{6} + \overset{2}{\cancel{4}} \frac{n(n+1)}{\cancel{2}}$$

$$= n(n+1)\left[\frac{2n+1}{6} + 2\right] = \frac{n(n+1)(2n+13)}{6}$$

Telescoping

(c) $$\sum_{k=1}^{100} \underbrace{\left(\frac{1}{k}-\frac{1}{k+1}\right)}_{a_k} = a_1 + a_2 + a_3 + \dots + a_{99} + a_{100}$$

$$= \frac{1}{1} - \frac{1}{1+1} + \frac{1}{2} - \frac{1}{2+1} + \frac{1}{3} - \frac{1}{3+1} + \dots$$

$$\dots + \frac{1}{99} - \frac{1}{99+1} + \frac{1}{100} - \frac{1}{100+1} = 1 - \frac{1}{101} = \frac{100}{101}$$

EXAMPLE 8. *Find the limit:* $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{n}\left[\left(\frac{j}{n}\right)^3+1\right]$

$$\sum_{j=1}^{n}\frac{1}{n}\left[\left(\frac{j}{n}\right)^3+1\right] = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j^3}{n^3}+1\right)$$

$$= \frac{1}{n}\left(\sum_{j=1}^{n}\frac{j^3}{n^3} + \sum_{j=1}^{n}1\right) = \frac{1}{n}\left[\frac{1}{n^3}\sum_{j=1}^{n}j^3 + n\right]$$

$$= \frac{1}{n^4}\sum_{j=1}^{n}j^3 + 1 \overset{\text{Theorem 6}}{=\!=} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 + 1$$

$$= \frac{n^2(n+1)^2}{4n^4} + 1 \xrightarrow[n\to\infty]{} \frac{1}{4}+1 = \boxed{\frac{5}{4}}$$