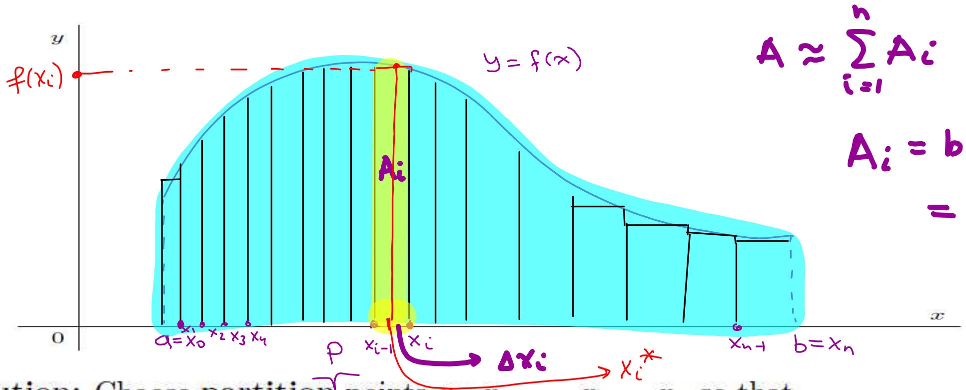
6.2: Area

Area problem: Let a function $f(x)$ be ~~positive~~ nonnegative on some interval $[a, b]$. Determine the area of the region between the function and the x-axis.



$$A \approx \sum_{i=1}^{n} A_i$$

$$A_i = \text{base} \cdot \text{height} = \Delta x_i \cdot f(x_i^*)$$

Solution: Choose **partition** P points $x_0, x_2, \ldots, x_{n-1}, x_n$ so that

$$a = x_0 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n = b.$$

Use notation $\Delta x_i = x_i - x_{i-1}$ for the length of ith subinterval $[x_{i-1}, x_i]$ $(1 \leq 1 \leq n)$

The length of the longest subinterval is denoted by $\|P\|$ (norm of P) $= \max\limits_{1 \leq i \leq n} \Delta x_i$

The location in each subinterval where we compute the height is denoted by x_i^*.

The area of the ith rectangle is

$$A_i = f(x_i^*) \, \Delta x_i$$

Then

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i$$

The area A of the region is:

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i = \int_a^b f(x) \, dx$$

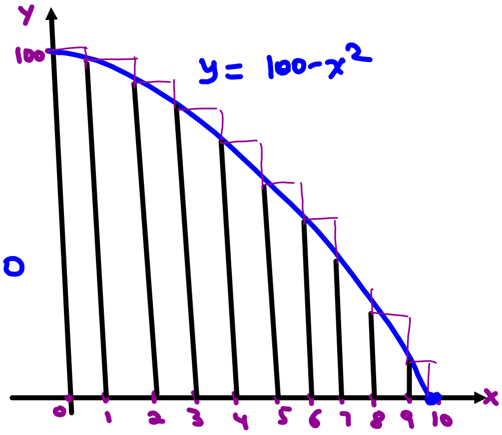
(see next section)

EXAMPLE 1. *Given* $f(x) = 100 - x^2$ *on* $[0, 10]$. *Let* $P = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *and* x_i^* *be left endpoint of* i*th subinterval.*

(a) *Find* $\|P\|$.

(b) *Find the sum of the areas of the approximating rectangles.*

(c) *Sketch the graph of* f *and the approximating rectangles.*



(a) $\|P\| = 1$, because $\Delta x_i = 1$ for $1 \le i \le 10$

(b) $A_i = f(x_i^*)\,\Delta x_i = f(i-1)\cdot 1$

$= 100 - (i-1)^2$

$= 100 - i^2 + 2i - 1 = 99 - i^2 + 2i$

$$\left.\begin{aligned} x_1^* &= 0 \\ x_2^* &= 1 \\ x_3^* &= 2 \\ &\vdots \\ x_{10}^* &= 9 \end{aligned}\right\} \Rightarrow x_i^* = i-1, \quad 1 \le i \le 10$$

$$A \approx \sum_{i=1}^{10} A_i = \sum_{i=1}^{10} (99 - i^2 + 2i)$$

$$= \sum_{i=1}^{10} 99 - \sum_{i=1}^{10} i^2 + 2\sum_{i=1}^{10} i \quad \text{Theorem 6 (Section 6.1)}$$

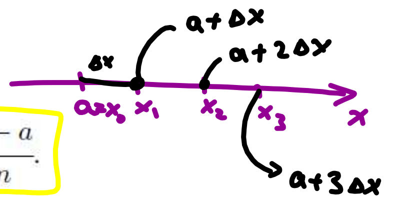
$$= 99 \cdot 10 - \frac{\cancel{10}^{5}(10+1)(2\cdot 10+1)}{\cancel{6}_{3}} + 2\,\frac{10(10+1)}{2}$$

$$= 990 - \frac{55\cdot \cancel{21}^{7}}{\cancel{3}} + 10\cdot 11 = 990 - 55\cdot 7 + 110$$

$$= 1100 - 385 = 715 \text{ units}^2$$

Riemann Sum for a function $f(x)$ on the interval $[a, b]$ is a sum of the form:

$$A \approx \sum_{i=1}^{n} A_i = \boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x_i.}$$



Consider a partition has equal subintervals: $\boxed{x_i = a + i\Delta x}$, where $\boxed{\Delta x = \frac{b-a}{n}}$.

LEFT-HAND RIEMANN SUM : $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x$

$x_1^* = x_0$

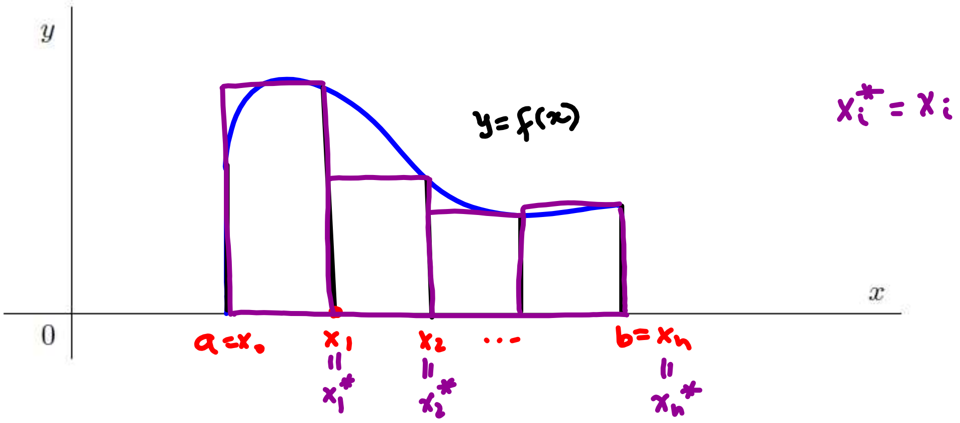
$x_2^* = x_1$

$x_3^* = x_2$

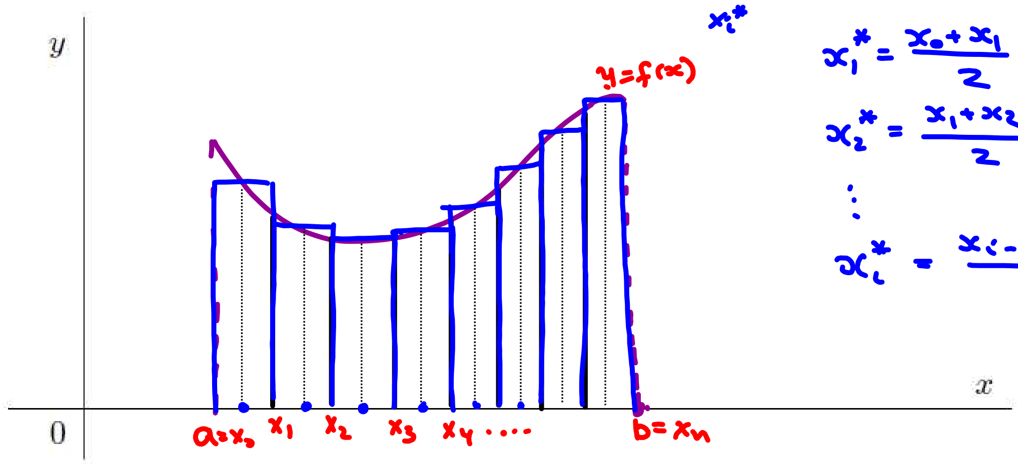
. . .

$x_i^* = x_{i-1} = a + (i-1)\Delta x$

RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$



MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\underbrace{\frac{x_i + x_{i-1}}{2}}_{x_i^*}\right)\Delta x = \Delta x \sum_{i=1}^{n} f\left(a + \frac{2i-1}{2}\Delta x\right)$



$$x_1^* = \frac{x_0 + x_1}{2}$$

$$x_2^* = \frac{x_1 + x_2}{2}$$

$$\vdots$$

$$x_i^* = \frac{x_{i-1} + x_i}{2}$$

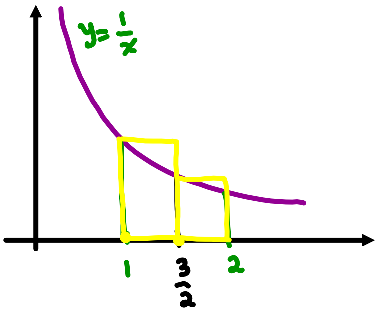
$$\frac{x_i + x_{i-1}}{2} = \frac{a + i\Delta x + a + (i-1)\Delta x}{2} = \frac{2a + \Delta x(i + i - 1)}{2}$$

$$= a + \frac{2i-1}{2}\Delta x$$

EXAMPLE 2. *Given* $f(x) = \frac{1}{x}$ *on* $[1, 2]$ ($a = 1$, $b = 2$). *Calculate* L_2, R_2, M_2. $\Rightarrow n = 2$

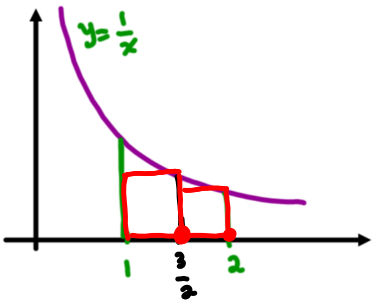
$\Delta x = \frac{b-a}{2}$

$\Delta x = \frac{2-1}{2} = \frac{1}{2}$



L_2 left-hand R.S.

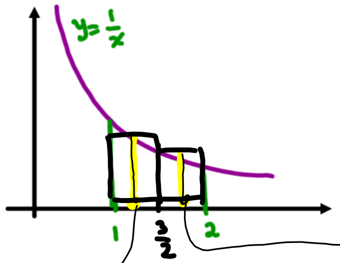
$$L_2 = f(1)\,\Delta x + f\left(\tfrac{3}{2}\right)\Delta x$$
$$= \left(f(1) + f\left(\tfrac{3}{2}\right)\right)\Delta x = \left(1 + \tfrac{2}{3}\right)\tfrac{1}{2} = \tfrac{5}{3}\cdot\tfrac{1}{2} = \boxed{\tfrac{5}{6}}$$



R_2 Right-hand R.S

$$R_2 = \left(f\left(\tfrac{3}{2}\right) + f(2)\right)\Delta x$$
$$= \left(\tfrac{2}{3} + \tfrac{1}{2}\right)\tfrac{1}{2} = \tfrac{7}{6}\cdot\tfrac{1}{2} = \boxed{\tfrac{7}{12}}$$

M_2 midpoint R.S.



$x_1^* = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4}$

$x_2^* = \frac{\frac{3}{2} + 2}{2} = \frac{7}{4}$

$$M_2 = \left(f(x_1^*) + f(x_2^*)\right)\Delta x$$
$$= \left(f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right)\tfrac{1}{2}$$
$$= \left(\tfrac{4}{5} + \tfrac{4}{7}\right)\tfrac{1}{2}$$
$$= {}^{2}\cancel{4}\left(\tfrac{1}{5} + \tfrac{1}{7}\right)\tfrac{1}{\cancel{2}_{\,1}}$$
$$= 2\cdot\tfrac{12}{35} = \boxed{\tfrac{24}{35}}$$

EXAMPLE 3. *Represent area bounded by* $f(x)$ *on the given interval using Riemann sum. Do not evaluate the limit.*

$$A = \lim_{\|P\| \to 0} \Delta x \sum_{i=1}^{n} f(x_i^*)$$

(a) $f(x) = x^2 + 2$ *on* $[0, 3]$ *using right endpoints.*

$a = 0$, $b = 3$ $\Rightarrow$ $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$\|P\| = \max_{1 \le i \le n} \Delta x_i = \Delta x = \frac{3}{n} \to 0 \iff n \to \infty$

$x_i^* = x_i = a + i\Delta x = 0 + i\frac{3}{n} = \frac{3i}{n}$

$f(x_i^*) = f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^2 + 2 = \frac{9i^2}{n^2} + 2$

$$A = \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 2 \right)$$

(b) $f(x) = \sqrt{x^2+2}$ *on* $[0,3]$ *using left endpoints.*

$$A = \lim_{\|P\| \to 0} \Delta x \sum_{i=1}^{n} f(x_i^*)$$

$\Delta x = \frac{3}{n}$

$\|P\| \to 0 \iff n \to \infty$

$x_i^* = x_{i-1} = a + (i-1)\Delta x = \frac{3(i-1)}{n}$

$f(x_i^*) = \sqrt{\left(\frac{3(i-1)}{n}\right)^2 + 2} = \sqrt{\frac{9(i-1)^2}{n^2} + 2}$

$$A = \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{\frac{9(i-1)^2}{n^2} + 2}$$

EXAMPLE 4. *The following limits represent the area under the graph of $f(x)$ on an interval $[a, b]$. Find $f(x), a, b$.*

(a) $\underbrace{\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}}_{R_n}$ $f(x_i^*)$ vs $\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*)$

$\Delta x = \frac{b-a}{n} = \frac{3}{n} \Rightarrow b-a=3$

$f(x) = \sqrt{1+x}$

$x_i^* = x_i = \frac{3i}{n} = a + i\Delta x = a + \frac{3i}{n} \Rightarrow a=0 \Rightarrow b=3$

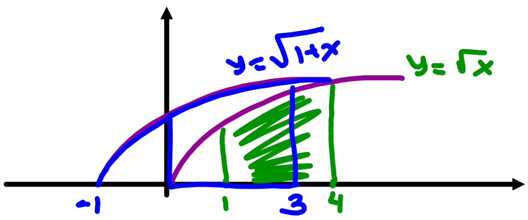
another solution

$f(x)=\sqrt{x}$

$x_i^* = x_i = 1+\frac{3i}{n} = a+\frac{3i}{n}$

$a=1, \quad b=4$

$$\boxed{f(x)=\sqrt{1+x}, \quad a=0, \ b=3}$$



(b) $\displaystyle\lim_{n\to\infty}\frac{10}{n}\sum_{i=1}^{n}\frac{1}{1+\Big(\underbrace{7+\frac{10i}{n}}_{x_i^*}\Big)^3}$ vs $\displaystyle\lim_{n\to\infty}\Delta x\sum_{n\to\infty}^{n} f(x_i^*)$

$\Delta x = \frac{b-a}{n} = \frac{10}{n} \Rightarrow b-a=10$

$f(x) = \frac{1}{1+x^3}$

$x_i^* = 7 + \frac{10i}{n} = x_i = a + i\Delta x = a + \frac{10i}{n}$

$\Rightarrow a = 7, \Rightarrow b = 17$

$\boxed{f(x) = \frac{1}{1+x^3}, \quad a=7, \quad b=17}$