6.3: The Definite Integral

DEFINITION 1. *The* **definite integral of** f **from** a **to** b *is*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. Here P is a partition of the interval $[a,b]$, $\Delta x_i =$ $x_i - x_{i-1}$ ~~$(b-a)/n$~~, and x_i^* is any point in the ith subinterval. If the limit does exist, then f is called **integrable** *on the interval $[a,b]$.*

EXAMPLE 2. *Express the limit as a definite integral:*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\underbrace{3\left(1+\frac{2i}{n}\right)^5 - 6}_{f(x_i^*)}\right] \underbrace{\frac{2}{n}}_{\Delta x} = \int_a^b (3x^5 - 6)\,dx$$

First identify $f(x)$, a, b.

$\Delta x = \frac{2}{n} = \frac{b-a}{n} \Rightarrow b - a = 2$

$f(x) = 3x^5 - 6$

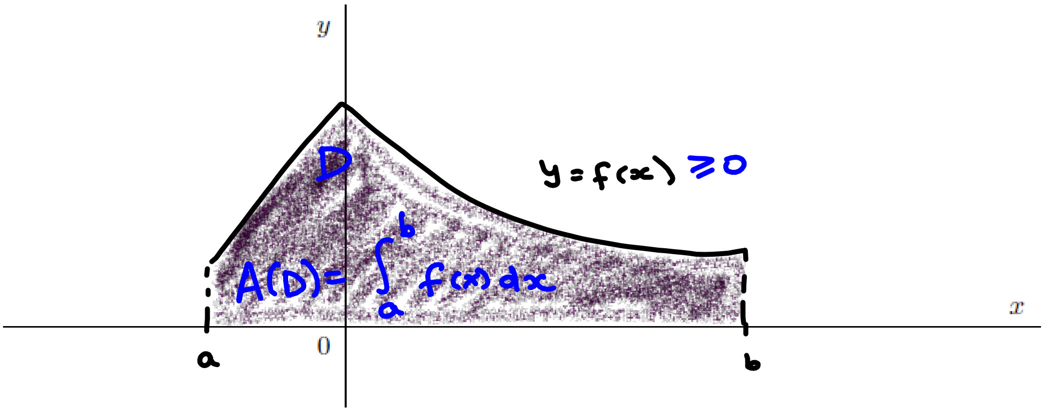
$x_i^* = \underbrace{1 + \frac{2i}{n}}$ $=$ $x_i = a + i\Delta x = \underbrace{a + \frac{2i}{n}} \Rightarrow a = 1$

Right-hand sum

$b = 3$

If $f(x) \geq 0$ on the interval $[a, b]$, then the definite integral is the area of region D bounded by the graph function f and the lines $y = 0$, $x = a$ and $x = b$.

$$D = \{(x,y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$$

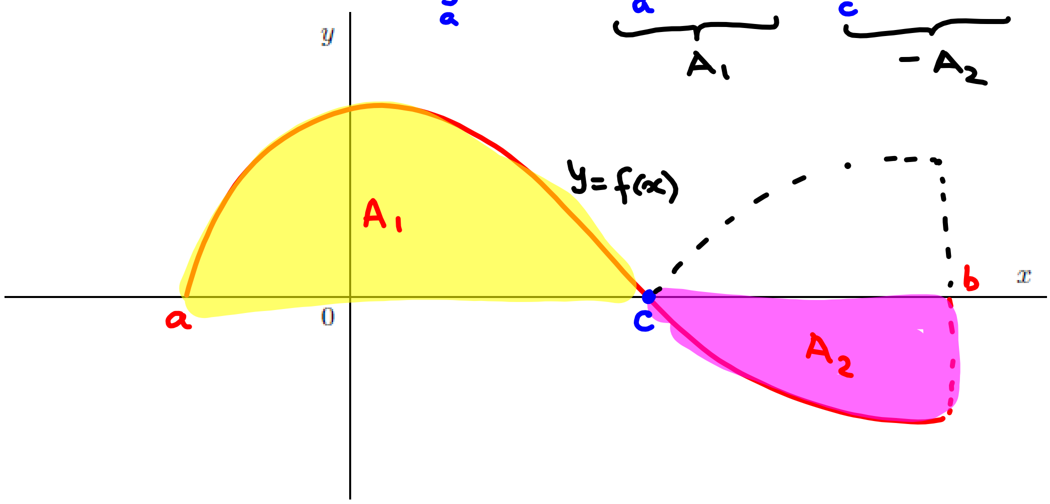


In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the region below the x and above the graph of f.

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{A_1} + \underbrace{\int_c^b f(x)\,dx}_{-A_2}$$



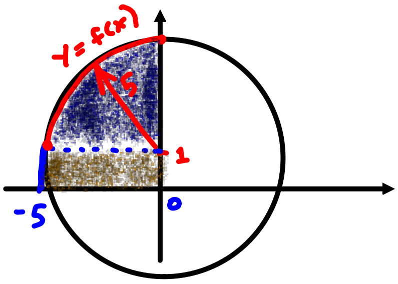
$$\int_a^b |f(x)|\,dx = \text{area below } y = |f(x)|$$

EXAMPLE 3. *Evaluate the following integrals by interpreting each in terms of areas:*

(a) $\displaystyle\int_{-5}^{0} \underbrace{(1+\sqrt{25-x^2})}\,dx$

$f(x) > 0$ on $[-5,0]$ $\Rightarrow$ the integral = area below the graph of $y = 1+\sqrt{25-x^2}$ on $[-5,0]$

$f(x) = 1+\sqrt{25-x^2}$



$y = 1+\sqrt{25-x^2}$

$(y-1)^2 = (\sqrt{25-x^2})^2, \quad y-1 \geq 0$

$(y-1)^2 = 25-x^2, \quad y \geq 1$

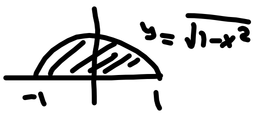
$x^2+(y-1)^2 = 25, \quad y\geq 1, \quad -5 \leq x \leq 0$

$\displaystyle\int_{-5}^{0}(1+\sqrt{25-x^2})\,dx = A_{\square} + A$

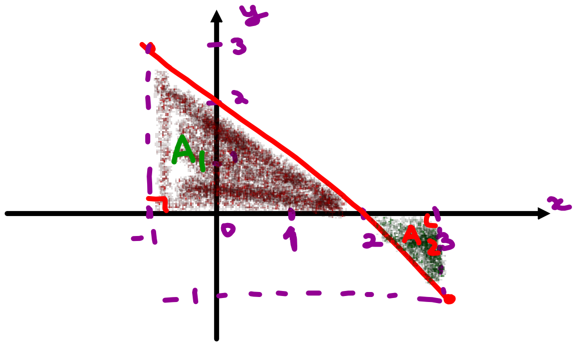
$= 5\cdot 1 + \dfrac{\pi 5^2}{4} = \boxed{5+\dfrac{25\pi}{4}}$

Remark Similarly,

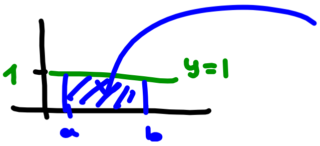
$$\int_{-1}^{1}\sqrt{1-x^2} = \frac{\pi}{2}$$



(b) $\int_{-1}^{3} \underbrace{(2-x)}_{f(x)=2-x}\, dx = A_1 - A_2 = \frac{3\cdot 3}{2} - \frac{1\cdot 1}{2} = \frac{9}{2} - \frac{1}{2} = \boxed{4}$



Properties of Definite Integrals:

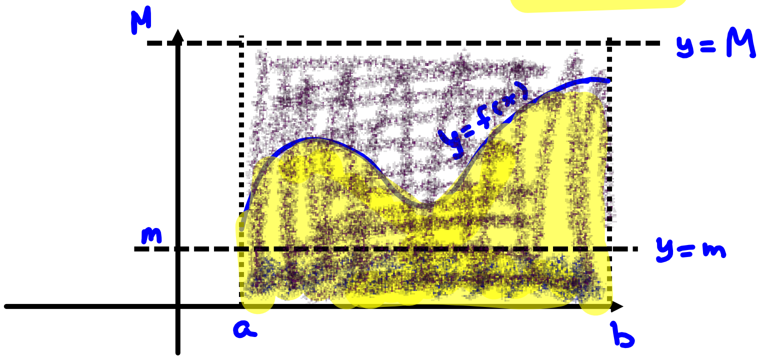


$\int_a^b dx = \int_a^b 1\,dx = (b-a)\cdot 1$

- $\displaystyle\int_a^b \mathrm{d}x = b - a$ Also length of $[a,b]$
- $\displaystyle\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where c is any constant

using limit's properties

- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \le c \le b$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \ge 0$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge 0$
- If $f(x) \ge g(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$
- If $m \le f(x) \le M$ for $a \le x \le b$, then $\displaystyle m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$.

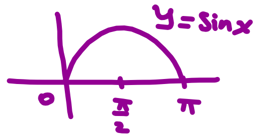


EXAMPLE 4. *Write as a single integral:*

$$\int_{3}^{5} f(x)\,\mathrm{d}x + \int_{0}^{3} f(x)\,\mathrm{d}x - \int_{6}^{5} f(x)\,\mathrm{d}x + \underbrace{\int_{5}^{5} f(x)\,\mathrm{d}x}_{=0}$$

$$= \int_{3}^{5} + \int_{0}^{3} + \int_{5}^{6} = \int_{0}^{3} + \int_{3}^{5} + \int_{5}^{6} = \int_{0}^{6} f(x)\,dx$$

EXAMPLE 5. *Estimate the value of* $\int_0^{\pi} (4\sin^5 x + 3)\,dx$



$-1 \le \sin x \le 1$ in general

in our case for $0 \le x \le \pi$ we have

$0 \le \sin x \le 1$

$0^5 \le \sin^5 x \le 1^5$

$0 \le \sin^5 x \le 1$

$0 \cdot 4 \le 4\sin^5 x \le 1 \cdot 4$

$0 \le 4\sin^5 x \le 4$

$0 + 3 \le 4\sin^5 x + 3 \le 4 + 3$

$\underbrace{3}_{m} \le 4\sin^5 x + 3 \le \underbrace{7}_{M}$

If $m \le f(x) \le M$ then

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$a = 0$, $b = \pi$

$b - a = \pi$

$$\boxed{3\pi \le \int_0^5 4\sin^5 x + 3 \le 7\pi}$$