6.4: The fundamental Theorem of Calculus

<u>The fundamental Theorem of Calculus :</u>

PART I If $f(x)$ is continuous on $[a, b]$ then $g(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

EXAMPLE 1. *Differentiate* $g(x) = \int_{-4}^{x} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t) \text{ cont and diff}}\, dt$

By FTC $\Rightarrow g'(x) = f(x) = e^{2x}\cos^2(1-5x)$

EXAMPLE 2. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Bigg(\underbrace{\int_a^{u(x)} f(t)\,\mathrm{d}t}_{g(u(x))}\Bigg) = f(u(x))u'(x).$$

$g(x) = \int_a^x f(t)\,dt \Rightarrow$

$\Downarrow$

$g'(x) = f(x)$

By Chain Rule

$$\frac{d}{dx}\left(g(u(x))\right) = g'(u)\cdot u'(x) \underset{\text{By FTC}}{=} f(u)\cdot u'(x)$$

Let $u(x)$ and $v(x)$ be differentiable functions and $f(x)$ be a continuous one. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x) - f(v(x))v'(x).$$

EXAMPLE 3. *Differentiate $g(x)$ if*

(a) $g(x) = \int_{-4}^{x^3} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t)}\,\mathrm{d}t$

$u(x) = x^3 \Rightarrow u'(x) = 3x^2$

$g'(x) = f(u)\,u' = e^{2x^3}\cos^2(1-5x^3)\cdot 3x^2$

$$\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{c} f(t)\,dt + \int_{c}^{u(x)} f(t)\,dt$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = \frac{d}{dx}\left(-\int_{c}^{v(x)} f(t)\,dt + \int_{c}^{u(x)} f(t)\,dt\right)$$

$$= -f(v(x))\,v'(x) + f(u(x))\,u'(x)$$

(b) $g(x) = \int_{e^{x^2}}^{1} \frac{t+1}{\ln t + 3}\,\mathrm{d}t = -\int_{1}^{e^{x^2}} \underbrace{\frac{t+1}{\ln t + 3}}_{f(t)}\,dt$

$u(x) = e^{x^2}$

$u'(x) = 2x\,e^{x^2}$

$$g'(x) = -f(u)\,u'$$

$$= -\frac{e^{x^2}+1}{\ln e^{x^2}+3} \cdot 2x\,e^{x^2} \quad \leftarrow \quad \ln e^a = a$$

$$= -\frac{2x\,e^{x^2}\left(e^{x^2}+1\right)}{x^2+3}$$

(c) $g(x) = \int_{x^2}^{\sin x} \frac{\cos t}{t}\,\mathrm{d}t$

$f(t) = \frac{\cos t}{t}$

$u(x) = \sin x \Rightarrow u'(x) = \cos x$

$v(x) = x^2 \Rightarrow v'(x) = 2x$

$g'(x) = f(u)\,u' - f(v)\,v'$

$= \underbrace{\frac{\cos(\sin x)}{\sin x}\,\cos x}_{= \cot x} - \frac{\cos x^2}{x^{\cancel{2}}} \cdot 2\cancel{x}$

$= \cot x \, \cos(\sin x) - \frac{2\cos x^2}{x}$

Fundamental Theorem of Calculus

PART II If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative for $f(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

Newton-Leibnitz formula

$$\left(F(x)+c\right)\Big|_a^b =$$

$$= F(b)+\cancel{c}-\left(F(a)+\cancel{c}\right)$$

$$= F(b)-F(a)$$

$$F'(x) = f(x)$$

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b)-F(a)$$

EXAMPLE 4. *Evaluate*

1. $\displaystyle\int_1^5 \frac{1}{x^2}\,\mathrm{d}x = \int_1^5 x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^5 = -\frac{1}{x}\Big|_1^5$

$$= -\left(\frac{1}{5}-\frac{1}{1}\right) = -\left(-\frac{4}{5}\right) = \frac{4}{5}$$

Question: $\displaystyle\int_{-1}^5 \frac{1}{x^2}\,dx$ Newton-Leibnitz formula is not applicable here because $\frac{1}{x^2}$ has infinite discontinuity at $x=0$.

2. $\displaystyle\int_{-\pi/2}^0 (\cos x - 4\sin x)\,\mathrm{d}x$

$$\left(\sin x - 4(-\cos x)\right)\Big|_{-\frac{\pi}{2}}^0 = \left(\sin x + 4\cos x\right)\Big|_{-\frac{\pi}{2}}^0$$

$$= \sin 0 + 4\cos 0 - \left(\sin\left(-\frac{\pi}{2}\right) + 4\cos\left(-\frac{\pi}{2}\right)\right)$$

$$= 0 + 4 - \left((-1) + 0\right) = 4+1 = \boxed{5}$$

3. $\int_0^1 (u^3+2)^2 \, \mathrm{d}u$

$(a+b)^2 = a^2 + 2ab + b^2$

$= \int_0^1 \left((u^3)^2 + 2 \cdot u^3 \cdot 2 + 2^2 \right) du$

$= \int_0^1 \left(u^6 + 4u^3 + 4 \right) du = \left(\frac{u^{6+1}}{6+1} + 4 \frac{u^{3+1}}{3+1} + 4u \right) \Big|_0^1$

$= \frac{1}{7} + 1 + 4 = \frac{36}{7}$

EXAMPLE 5. *Evaluate*

1. $\displaystyle\int_1^2 \frac{2x^5 - x + 3}{x^2}\,dx = \int_1^2 \left(\frac{2x^5}{x^2} - \frac{x}{x^2} + \frac{3}{x^2}\right) dx$

$$= \int_1^2 \left(2x^3 - \frac{1}{x} + 3x^{-2}\right) dx$$

$$= \left(\frac{2x^{3+1}}{3+1} - \ln|x| + \frac{3x^{-2+1}}{-2+1}\right)\Bigg|_1^2$$

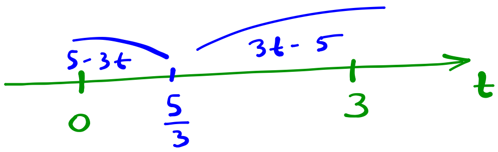
$$= \left(\frac{x^4}{2} - \ln x - \frac{3}{x}\right)\Bigg|_1^2 = \frac{2^4}{2} - \ln 2 - \frac{3}{2} - \left(\frac{1}{2} - \ln 1 - 3\right)$$

$$= 8 - \ln 2 \underbrace{- \frac{3}{2} - \frac{1}{2}}_{-2} + 0 + 3 = \boxed{9 - \ln 2}$$

2. $\displaystyle\int_0^3 |3t-5|\,dt$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

$$|3t-5| = \begin{cases} 3t-5, & \text{if } 3t-5 \geq 0 \\ -(3t-5), & \text{if } 3t-5 \leq 0 \end{cases} = \begin{cases} 3t-5 & \text{if } t \geq \frac{5}{3} \\ 5-3t & \text{if } t \leq \frac{5}{3} \end{cases}$$



$$\int_0^3 |3t-5|\,dt = \int_0^{5/3} |3t-5|\,dt + \int_{5/3}^3 |3t-5|\,dt$$

$$= \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$$

$$= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3$$

$$= \underbrace{\underbrace{\left(\frac{25}{3} - \frac{25}{6}\right)}_{\frac{25}{6}} - 0 + \frac{27}{2} - 15 - \underbrace{\left(\frac{25}{6} - \frac{25}{3}\right)}_{-\frac{25}{6}}}_{2 \cdot \frac{25}{6} = \frac{25}{3}}$$

$$= \frac{25}{3} + \underbrace{\frac{27}{2} - 15}_{-\frac{3}{2}} = \frac{25}{3} - \frac{3}{2} = \frac{50-9}{6} = \frac{41}{6}$$

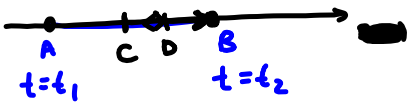
Applications of the Fundamental Theorem

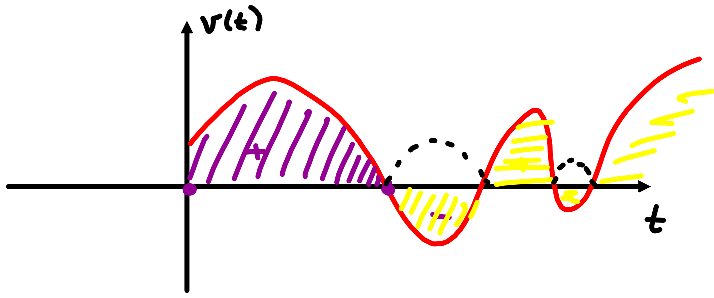
If a particle is moving along a straight line then application of the Fundamental Theorem to $s'(t) = v(t)$ yields:

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1) = \text{displacement}.$$

Show that

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t.$$





EXAMPLE 6. *A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled by the particle during the time period $1 \le t \le 6$.*

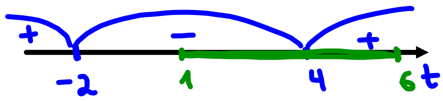
$$\text{Displacement} = \int_1^6 v(t)\,dt = \int_1^6 (t^2-2t-8)\,dt = \left(\frac{t^3}{3}-t^2-8t\right)\Big|_1^6$$

$$= 72-36-48-\left(\frac{1}{3}-1-8\right) = -12-\frac{1}{3}+9$$

$$= -3-\frac{1}{3} = -\frac{10}{3}$$

$$\text{Total Distance Traveled} = \int_1^6 |v(t)|\,dt = \int_1^6 |t^2-2t-8|\,dt$$

$$t^2-2t-8 = (t-4)(t+2) = \begin{cases} t^2-2t-8, & 4 \le t \le 6 \\ -(t^2-2t-8), & 1 \le t \le 4 \end{cases}$$



$$\text{TDT} = -\int_1^4 (t^2-2t-8)\,dt + \int_4^6 (t^2-2t-8)\,dt$$

$$= -\left(\frac{t^3}{3}-t^2-8t\right)\Big|_1^4 + \left(\frac{t^3}{3}-t^2-8t\right)\Big|_4^6$$

$$= -\left(\frac{64}{3}-16-32-\left(\frac{1}{3}-1-8\right)\right)$$

$$+\left(\frac{216}{3}-36-48-\left(\frac{64}{3}-16-32\right)\right) = \dots = \boxed{\frac{98}{3}}$$