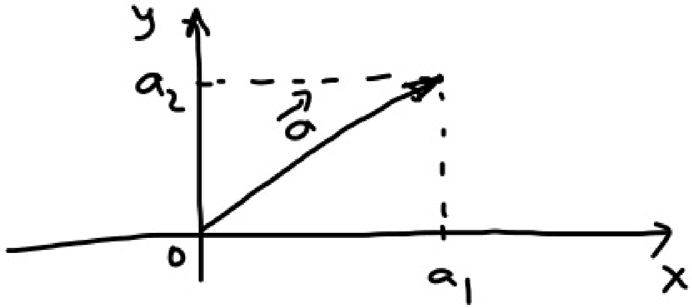
Section 1.1: Vectors

Quantities that we measure that have magnitude but not direction are called **scalars**.

DEFINITION 1. *A* ***vector*** *is a quantity that has both magnitude and direction. A 2-dimensional vector is an ordered pair* $\mathbf{a} = \langle a_1, a_2 \rangle$ *. The numbers* a_1 *and* a_2 *are called the* ***components*** *of the vector* $\mathbf{a}$.

$$\vec{a} = \langle a_1, a_2 \rangle$$



Typical notation to designate a vector is a boldfaced character or a character with and arrow on it (i.e. $\mathbf{a}$ or $\vec{a}$).

DEFINITION 2. *Given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector* **a** *with representation $\overrightarrow{AB}$ is*

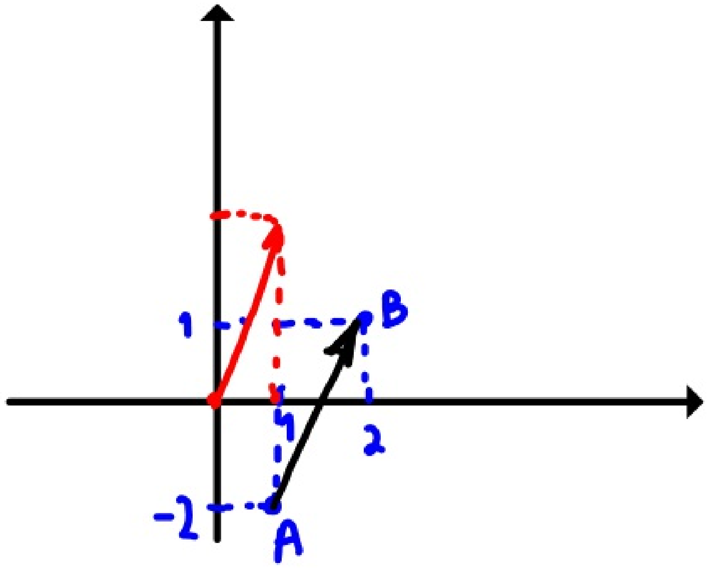
$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle .$$

The point A here is initial point and B is terminal one.

A vector with the initial point located at the origin is called the **position** vector (or we say that a vector is in standard position).

Vectors are equal if they have the same length and direction (same slope).

EXAMPLE 3. *Graph the vector with initial point $A(1, -2)$ and terminal point $B(2,1)$. Find the components of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.*



$$\overrightarrow{AB} = B - A = \langle 2,1\rangle - \langle 1,-2\rangle =$$

$$= \langle 2-1, 1-(-2)\rangle = \langle 1,3\rangle$$

$$\overrightarrow{BA} = \langle 1,-2\rangle - \langle 2,1\rangle =$$

$$= \langle 1-2, -2-1\rangle = \langle -1,-3\rangle =$$

$$= -\langle 1,3\rangle = -\overrightarrow{AB}$$

Vector operations

c is a real number

$-2\langle 1,3\rangle = \langle -2\cdot 1, -2\cdot 3\rangle = \langle -2,-6\rangle$

- *Scalar Multiplication*: If c is a scalar and $\mathbf{a} = \langle a_1, a_2\rangle$, then

$$c\mathbf{a} = c\langle a_1, a_2\rangle = \langle ca_1, ca_2\rangle .$$

DEFINITION 4. *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are called* **parallel** *if* $\mathbf{b} = c\mathbf{a}$ *with some scalar* c.

If $c > 0$ then a and ca have the same direction, if $c < 0$ then a and ca have the opposite direction.

$\vec{a} = \langle a_1, a_2\rangle$

$\vec{b} = \langle b_1, b_2\rangle$

$\vec{b} = c\vec{a}$

$c > 0$

$c < 0$

$\langle b_1, b_2\rangle = \langle ca_1, ca_2\rangle$

$b_1 = ca_1, \quad b_2 = ca_2 \Rightarrow c = \boxed{\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2}}$

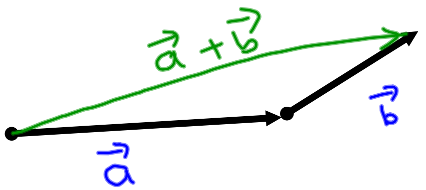
- *Vector addition*: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then

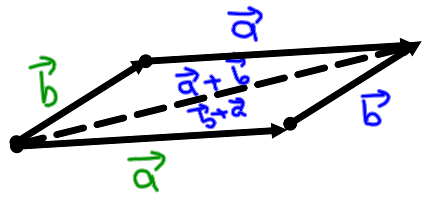
$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle .$$

$\langle -1, 0 \rangle + \langle 3, 4 \rangle$
$= \langle -1+3, 0+4 \rangle$
$= \langle 2, 4 \rangle$

Triangle Law **Parallelogram Law**

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$





$\mathbf{a} + \mathbf{b}$ is called the **resultant vector**

EXAMPLE 5. *Let* $\mathbf{a} = \langle -1, 2 \rangle$ *and* $\mathbf{b} = \langle 2.1, -0.5 \rangle$. *Then* $3\mathbf{a} + 2\mathbf{b} =$

$3\vec{a} + 2\vec{b} = 3\langle -1, 2 \rangle + 2\langle 2.1, -0.5 \rangle$

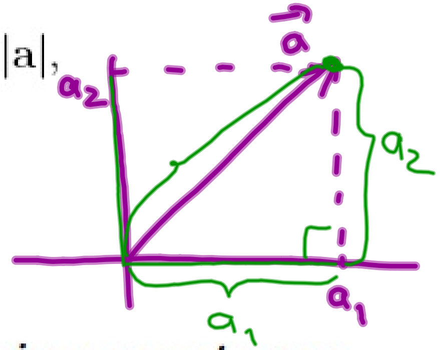
$= \langle 3\cdot(-1), 3\cdot 2 \rangle + \langle 2\cdot 2.1, 2\cdot(-0.5) \rangle$

$= \langle -3, 6 \rangle + \langle 4.2, -1 \rangle = \langle -3+4.2, 6+(-1) \rangle$

$= \langle 1.2, 5 \rangle$

The **magnitude** or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $|\mathbf{a}|$,

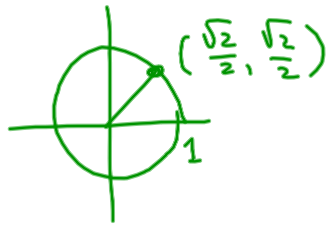
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$



EXAMPLE 6. *Find magnitudes of the following vectors:*

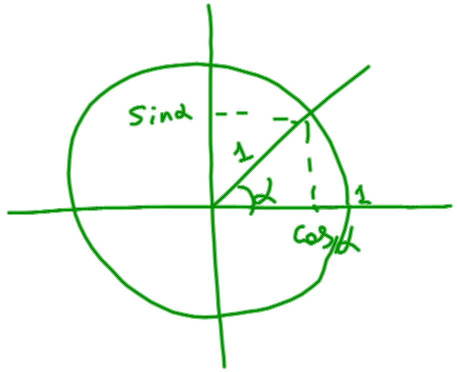
(a) $|\langle 3, -8 \rangle| = \sqrt{3^2 + (-8)^2} = \sqrt{9 + 64} = \sqrt{73}$

(b) $\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle\right| = 1$



(c) $|\mathbf{0}| = |\langle 0, 0 \rangle| = \sqrt{0^2 + 0^2} = 0$

(d) $|\langle \cos\alpha, \sin\alpha \rangle| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

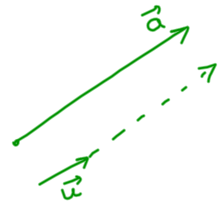


A **unit** vector is a vector with length one. Any vector can be made into a unit vector by dividing it by its length. So, a unit vector in the direction of **a** is

by normalizing

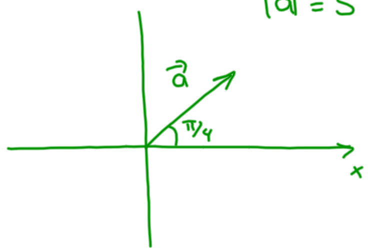
$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{|\vec{a}|} \cdot \langle a_1, a_2 \rangle = \left\langle \frac{a_1}{|a|}, \frac{a_2}{|a|} \right\rangle$$

Any vector **a** can be fully represented by providing its length, $|\mathbf{a}|$ and a unit vector **u** in its direction:



$$\boxed{\mathbf{a} = |\mathbf{a}|\,\mathbf{u}}$$

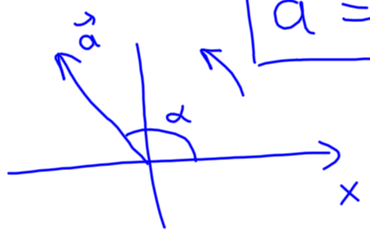
$|\vec{a}| = 5$



$$\vec{a} = |\vec{a}| \cdot \vec{u} = 5 \underbrace{\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle}_{\left\langle \cos\frac{\pi}{4},\, \sin\frac{\pi}{4} \right\rangle} = \left\langle \frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2} \right\rangle$$

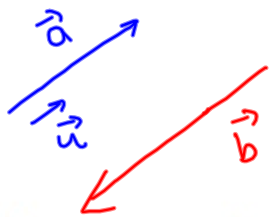
In general if $\vec{a}$ makes an angle α with positive direction of the x-axis in counterclockwise direction, then

$$\boxed{\vec{a} = |\vec{a}| \langle \cos\alpha, \sin\alpha \rangle = \langle |\vec{a}| \cos\alpha, |a| \sin\alpha \rangle}$$



(a) *a unit vector that has the same direction as* **a**;

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, -1 \rangle}{\sqrt{2^2 + (-1)^2}} = \frac{\langle 2, -1 \rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$



(b) *a vector* **b** *in the direction opposite to* **a** *s.t* $|\mathbf{b}| = 7$.

such that

$$\vec{b} = |\vec{b}| \cdot (-\vec{u}) = 7 \cdot \left(- \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle \right)$$

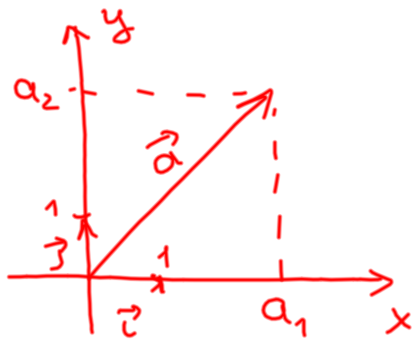
$$= 7 \cdot \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = \left\langle -\frac{14}{\sqrt{5}}, \frac{7}{\sqrt{5}} \right\rangle$$

The **standard basis vectors** are given by the unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ along the x and y directions, respectively. Using the basis vectors, one can represent any vector $\mathbf{a} = \langle a_1, a_2 \rangle$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

$|\vec{i}| = |\langle 1, 0 \rangle| = 1$

$|\vec{j}| = |\langle 0, 1 \rangle| = 1$



$$\vec{a} = a_1\vec{i} + a_2\vec{j} = a_1\langle 1, 0 \rangle + a_2\langle 0, 1 \rangle$$

$$= \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = \langle a_1, a_2 \rangle$$

EXAMPLE 8. *Given* $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \langle 5, -2 \rangle$. *Find a scalars* s *and* t *such that* $s\mathbf{a} + t\mathbf{b} = -4\mathbf{j}$.

$\langle 2, -1 \rangle$ $\quad$ $\langle 0, 1 \rangle$

$$s\vec{a} + t\vec{b} = -4\langle 0, 1 \rangle$$

$$s\langle 2, -1 \rangle + t\langle 5, -2 \rangle = \langle 0, -4 \rangle$$

$$\langle 2s, -s \rangle + \langle 5t, -2t \rangle = \langle 0, -4 \rangle$$

$$\langle 2s + 5t, -s - 2t \rangle = \langle 0, -4 \rangle$$

$$\begin{cases} 2s + 5t = 0 \\ -s - 2t = -4 \quad \times 2 \end{cases} \Rightarrow + \begin{array}{r} 2s + 5t = 0 \\ -2s - 4t = -8 \\ \hline t = -8 \end{array}$$

$$\Downarrow$$

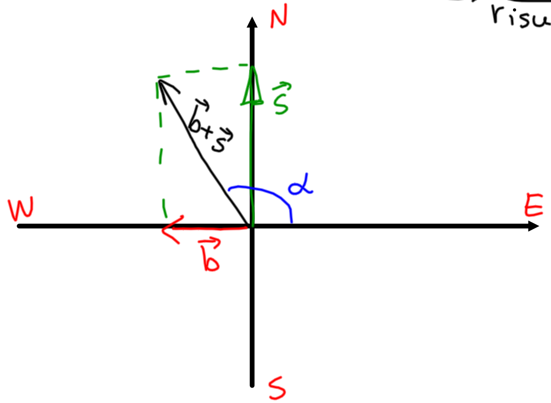
$$s = 4 - 2t = 4 - 2\cdot(-8) = 20$$

Answer: $t = -8$, $s = 20$.

Applications: Quantities such as force, displacement or velocity that have direction as well as magnitude are represented by vectors.

EXAMPLE 9. *Ben walks due West on the deck of a ship at 5 mph. The ship is moving north at a speed of 25 mph. Find the direction and speed of Ben relative to the surface of the water.*

risultant vector of $\vec{b}$ and $\vec{s}$



$|\vec{b}| = 5$

$\vec{b} = -5\vec{i} = \langle -5, 0 \rangle$

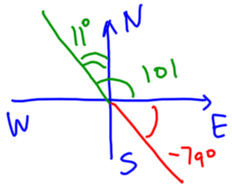
$|\vec{s}| = 25$

$\vec{s} = 25\vec{j} = \langle 0, 25 \rangle$

$$\underbrace{\vec{r}} = \vec{b} + \vec{s} = \langle -5, 0 \rangle + \langle 0, 25 \rangle = \langle -5, 25 \rangle = 5\langle -1, 5 \rangle$$

$$|\vec{r}| = |5\langle -1, 5 \rangle| = 5|\langle -1, 5 \rangle| = 5\sqrt{(-1)^2 + 5^2} = \underbrace{5\sqrt{26}}_{\text{speed}} \text{ mph}$$

$$\tan\alpha = \frac{5}{-1} = -5 \quad \Rightarrow \quad \alpha = -79^\circ + 180^\circ = \boxed{101^\circ}$$



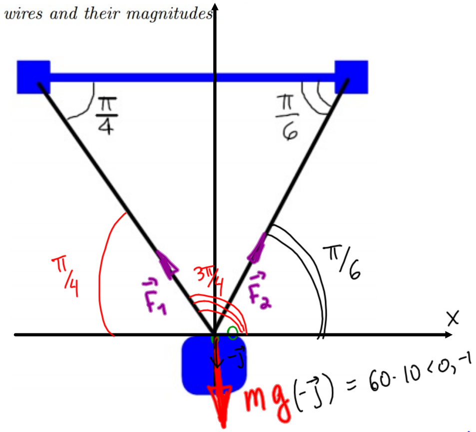
$\tan\alpha = \tan(\alpha \pm \pi)$

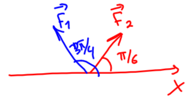
direction: N 11° W

11° west of the north direction.

EXAMPLE 10. *An 60 kg weight hangs from two wires as shown. Find the tensions (forces) in both wires and their magnitudes.*

$g \approx 10\ m/s^2$





$mg(-\vec{j}) = 60 \cdot 10 \langle 0,-1 \rangle = \langle 0,-600 \rangle$

First note that the given weight is in equilibrium, so

$$\vec{F}_1 + \vec{F}_2 + \langle 0,-600 \rangle = \langle 0,0 \rangle,$$

where

$$\vec{F}_1 = \left\langle |\vec{F}_1| \cos\tfrac{3\pi}{4},\ |\vec{F}_1| \sin\tfrac{3\pi}{4} \right\rangle = \left\langle -\frac{|\vec{F}_1|\sqrt{2}}{2},\ \frac{|\vec{F}_1|\sqrt{2}}{2} \right\rangle$$

$$\vec{F}_2 = \left\langle |\vec{F}_2| \cos\tfrac{\pi}{6},\ |\vec{F}_2| \sin\tfrac{\pi}{6} \right\rangle = \left\langle \frac{|\vec{F}_2|\sqrt{3}}{2},\ \frac{|\vec{F}_2|}{2} \right\rangle$$

$$\left\langle -\frac{|\vec{F}_1|\sqrt{2}}{2} + \frac{|\vec{F}_2|\sqrt{3}}{2} + 0,\ \frac{|\vec{F}_1|\sqrt{2}}{2} + \frac{|\vec{F}_2|}{2} - 600 \right\rangle = \langle 0,0 \rangle$$

$$\begin{cases} -\dfrac{|\vec{F}_1|\sqrt{2}}{2} + \dfrac{|\vec{F}_2|\sqrt{3}}{2} = 0 \quad (1) \\ + \\ \dfrac{|\vec{F}_1|\sqrt{2}}{2} + \dfrac{|\vec{F}_2|}{2} = 600 \end{cases}$$

$$0 + \frac{|\vec{F}_2|\sqrt{3}}{2} + \frac{|\vec{F}_2|}{2} = 600 \quad (\times 2)$$

$$|\vec{F}_2|(\sqrt{3}+1) = 1200$$

$$|\vec{F}_2| = \frac{1200}{(\sqrt{3}+1)} \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \frac{1200(\sqrt{3}-1)}{\underbrace{(\sqrt{3})^2 - 1^2}_{2}}$$

$$|\vec{F}_2| = 600(\sqrt{3}-1)$$

(1) implies

$$\frac{|\vec{F}_1|\sqrt{2}}{\cancel{2}} = \frac{|\vec{F}_2|\sqrt{3}}{\cancel{2}}$$

$$|\vec{F}_1| = \frac{|\vec{F}_2|\sqrt{3}}{\sqrt{2}} = \frac{\overset{300}{\cancel{600}}(\sqrt{3}-1)\sqrt{3}}{\cancel{\sqrt{2}}}\frac{\sqrt{2}}{\cancel{\sqrt{2}}}$$

$$= 300\sqrt{6}(\sqrt{3}-1)$$

Answer:

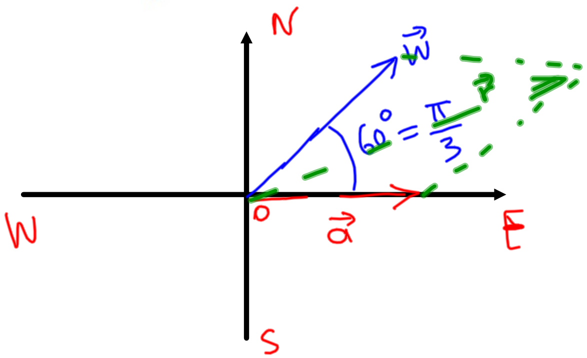
$$\vec{F}_1 = \left\langle -300\overset{\sqrt{3}}{\cancel{\sqrt{6}}}(\sqrt{3}-1)\cancel{\frac{\sqrt{2}}{2}},\ 300\overset{\sqrt{3}}{\cancel{\sqrt{6}}}(\sqrt{3}-1)\cancel{\frac{\sqrt{2}}{2}}\right\rangle$$

$$\boxed{\vec{F}_1 = \langle -300\sqrt{3}(\sqrt{3}-1),\ 300\sqrt{3}(\sqrt{3}-1)\rangle}$$

$$\vec{F}_2 = \left\langle 600(\sqrt{3}-1)\frac{\sqrt{3}}{2},\ \frac{600(\sqrt{3}-1)}{2}\right\rangle$$

$$\boxed{\vec{F}_2 = \langle 300(\sqrt{3}-1)\sqrt{3},\ 300(\sqrt{3}-1)\rangle}$$

EXAMPLE 11. *An airplane, flying due east at an airspeed of 450mph, encounters a 50-mph wind acting in the direction of E60^{o}N (60^o North of East). The airplane holds its compass heading due East but, because of the wind, acquires a new ground speed (i.e. the magnitude of the resultant) and direction. What are they?*



$\vec{a} = 450\,\vec{i} = \langle 450, 0\rangle$

$|\vec{w}| = 50$ mph

$\vec{w} = 50\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\rangle$

$\vec{w} = \langle 25, 25\sqrt{3}\rangle$

the resultant $\vec{r} = \vec{a} + \vec{w} = \langle 450, 0\rangle + \langle 25, 25\sqrt{3}\rangle$

$= \langle 475, 25\sqrt{3}\rangle = 25\langle 19, \sqrt{3}\rangle$

$|\vec{r}| = |25\langle 19, \sqrt{3}\rangle| = 25|\langle 19, \sqrt{3}\rangle|$

$= 25\sqrt{19^2 + (\sqrt{3})^2} = 25\sqrt{361+3} = 25\sqrt{364}$

$= 25\sqrt{91\cdot 4} = 25\cdot 2\sqrt{91} = 50\sqrt{91}$ mph

direction of $\vec{r}$: $\tan\alpha = \frac{\sqrt{3}}{19} \Rightarrow \alpha = 5.2^\circ$

Answer: speed is $50\sqrt{91}$ mph

direction: E 5.2° N