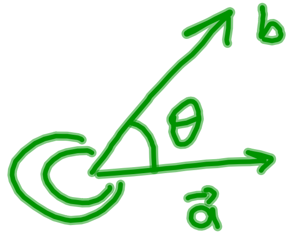
Section 1.2: The Dot Product



Let's start with two equivalent definitions of dot product.

DEFINITION 1. *The* **dot product** *of two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is the number*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, $0 \le \theta \le \pi$. *If either* $\mathbf{a}$ *or* $\mathbf{b}$ *is* $\mathbf{0}$, *then we define* $\mathbf{a} \cdot \mathbf{b} = 0$.

DEFINITION 2. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$ *is the number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Note that the formula from Definition 1 is often used not to compute a dot product but instead to find the angle between two vectors. Indeed, it implies:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\,\sqrt{b_1^2 + b_2^2}},$$

$0 \le \theta \le \pi$

EXAMPLE 3. *Given* a $= \langle 2, -3\rangle$ *and* b $= \langle 3, -4\rangle$.

(a) *Compute the dot product of* a *and* b.

$$\vec{a}\cdot\vec{b} = 2\cdot 3 + (-3)\cdot(-4) = 18$$

(b) *Determine the angle between* a *and* b.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{18}{\sqrt{2^2+(-3)^2}\ \sqrt{3^2+(-4)^2}} = \frac{18}{\sqrt{13}\cdot 5}$$

$$\theta \approx 3.18^\circ$$

Note that

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

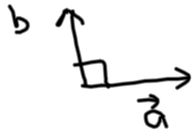
$$\vec{0} \cdot \vec{\mathbf{a}} = 0$$

$$\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \underbrace{\cos\theta}_{\cos 0 = 1} = |\vec{a}|^2$$

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2}$$

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

Two nonzero vectors a and b are orthogonal if and only if a · b = 0. (*Prove it!*)



$\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$, $\quad \vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0$

$$\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\underbrace{\cos\frac{\pi}{2}}_{=0} = 0$$

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow \underbrace{|\vec{a}|}_{\neq 0}\cdot\underbrace{|\vec{b}|}_{\neq 0}\cdot\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a}\perp\vec{b}$$

EXAMPLE 4. *Determine whether the given vectors are orthogonal, parallel, or neither.*

(a) $\langle 3,4\rangle$, $\langle -8,6\rangle$

$\langle 3,4\rangle \cdot \langle -8,6\rangle = 3\cdot(-8)+4\cdot 6 = 0 \Rightarrow$ orthogonal

$\cos\theta = \pm 1 \Rightarrow \vec{a} \parallel \vec{b}$

$\cos\theta = 0 \Rightarrow \vec{a} \perp \vec{b}$

(b) $\langle -7,-4\rangle$, $\langle 28,16\rangle$

$\langle -7,-4\rangle \parallel \langle 28,16\rangle$ parallel

$$\frac{-7}{28} = \frac{-4}{16}$$

(c) $\langle 1,1\rangle$, $\langle 2,3\rangle$ neither

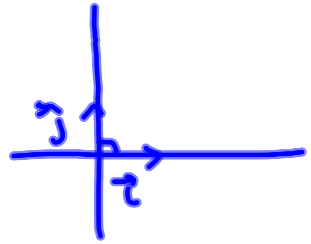
$\frac{1}{2} \neq \frac{1}{3} \Rightarrow$ not parallel

$\langle 1,1\rangle \cdot \langle 2,3\rangle = 1\cdot 2 + 1\cdot 3 \neq 0 \Rightarrow$ not orthogonal

EXAMPLE 5. *What is the dot product of* 12**j** *and* 11**i**?

$\vec{i} \perp \vec{j} \Rightarrow \vec{i}\cdot\vec{j} = 0 \Rightarrow (12\cdot 11)\vec{i}\cdot\vec{j} = 0$

$11\vec{i} \cdot 12j = 0$

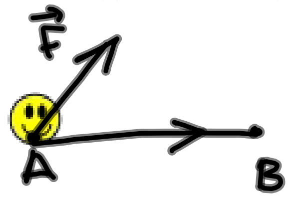


$12\vec{j} \cdot 11\vec{i} = \langle 0, 12\rangle \cdot \langle 11, 0\rangle = 0\cdot 11 + 12\cdot 0 = 0$

Conclusion: $s\vec{j} \perp t\vec{i}$ for all s,t.

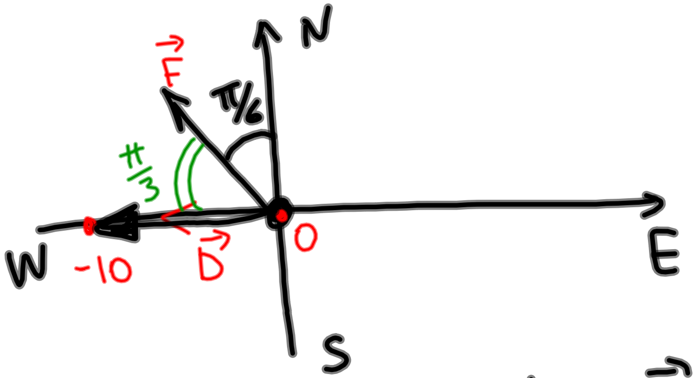
DEFINITION 6. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point A to point B is given by*

$$W = \mathbf{F} \cdot \mathbf{D} = \vec{F} \cdot \overrightarrow{AB}$$



where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

EXAMPLE 7. *Find the work done by a force of 50lb acting in the direction $N30^oW$ in moving an object 10ft due west.*



$|\vec{F}| = 50\ lb$

$\theta = \angle \vec{F}, \vec{D} = \frac{\pi}{3}$

$|\vec{D}| = 10$

$$W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cos\theta$$

$$= 50 \cdot 10 \cdot \cos\frac{\pi}{3} = 500 \cdot \frac{1}{2} = 250 \text{ ft-lb}$$

EXAMPLE 8. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1,1)$ *to* $B(5,6)$*. Find the work done if the distance is measured in meters*

$$\vec{D} = \vec{AB} = \langle 5,6\rangle - \langle 1,1\rangle = \langle 4,5\rangle$$

$$W = \vec{F}\cdot\vec{D} = \langle 25,4\rangle \cdot \langle 4,5\rangle$$

$$= 25\cdot 4 + 4\cdot 5 = 120\ \mathrm{J}$$

DEFINITION 9. *The* **orthogonal compliment** *of* $\mathbf{a} = \langle a_1, a_2 \rangle$ *is* $\boxed{\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle.}$

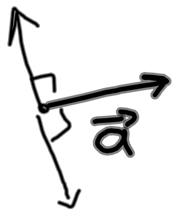
Note that $|\mathbf{a}| = |\mathbf{a}^{\perp}|$ and $\mathbf{a} \cdot \mathbf{a}^{\perp} = \langle a_1, a_2 \rangle \cdot \langle -a_2, a_1 \rangle$

$$\sqrt{a_1^2 + a_2^2} = \sqrt{(-a_2)^2 + a_1^2}$$

$$= -a_1 a_2 + a_2 a_1 = 0$$

$$\Downarrow$$

$$\vec{a} \perp \vec{a}^{\perp}$$



EXAMPLE 10. *Given* $\langle 4, -2 \rangle$, $\langle 2, -1 \rangle$, $\langle -2, 1 \rangle$ *and* $\mathbf{a} = \langle 1, 2 \rangle$. *Which of these vectors is*

- *orthogonal to* $\mathbf{a}$?

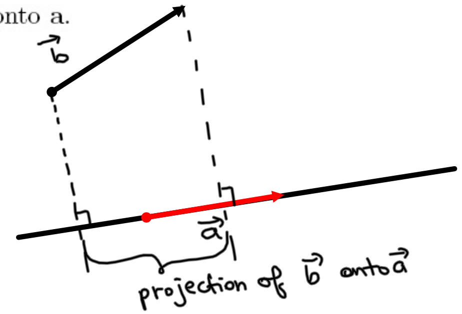
$\langle -2, 1 \rangle = \vec{a}^{\perp} \Rightarrow \langle -2, 1 \rangle \perp \vec{a}$

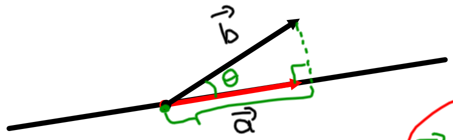
Since all three given vectors are parallel, all of them are orthogonal to $\vec{a}$.

- *the orthogonal compliment of* $\mathbf{a}$?

$\vec{a}^{\perp} = \langle -2, 1 \rangle$

Scalar and vector projections: For given two vectors **a** and **b** we determine the projection of **b** onto **a**.





$$\text{projection} = |\vec{b}| \cos\theta = \cancel{|\vec{b}|}\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,\cancel{|\vec{b}|}}$$

- The vector projection of **b** onto **a** is denotes by $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of **b** onto **a** (or the component of **b** along **a**) is denotes by $\mathrm{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

$$\mathrm{proj}_{\vec{a}}\,\vec{b} = \mathrm{comp}_{a}\vec{b}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$$

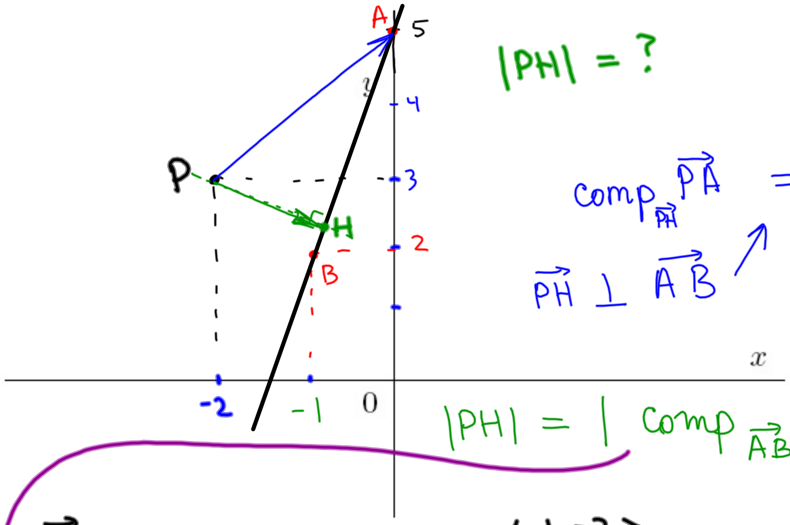
Remark

$$\mathrm{comp}_{2\vec{a}}\,\vec{b} = \mathrm{comp}_{\vec{a}}\,\vec{b}$$

EXAMPLE 11. *Given* $\mathbf{a} = \langle 4, 3\rangle$ *and* $\mathbf{b} = \langle 1, -1\rangle$. *Find:*

- $\mathbf{a} \cdot \mathbf{b} =$ $\langle 4, 3\rangle \cdot \langle 1, -1\rangle = 4 - 3 = 1$ $= \vec{b} \cdot \vec{a}$
- $|\mathbf{a}| =$ $\sqrt{4^2 + 3^2} = 5$
- $|\mathbf{b}| =$ $\sqrt{1^2 + (-1)^2} = \sqrt{2}$
- $\mathrm{proj}_{\mathbf{b}}\mathbf{a} =$ $\dfrac{\vec{b} \cdot \vec{a}}{|\vec{b}|^2} \cdot \vec{b} = \dfrac{1}{(\sqrt{2})^2} \cdot \langle 1, -1\rangle = \langle \frac{1}{2}, -\frac{1}{2} \rangle$
- $\mathrm{comp}_{\mathbf{a}}\mathbf{b} =$ $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{1}{5}$

EXAMPLE 12. *Find the distance from the point $P(-2,3)$ to the line $y = 3x + 5$.*



$|PH| = ?$

	A	B
x	0	-1
y	5	2

$\text{comp}_{\vec{PH}} \vec{PA} = \text{comp}_{\vec{AB}^{\perp}} \vec{PA}$

$\vec{PH} \perp \vec{AB}$

$|PH| = \left| \text{comp}_{\vec{AB}^{\perp}} \vec{PA} \right|$

$\vec{AB} = \langle -1, 2\rangle - \langle 0, 5\rangle = \langle -1, -3\rangle$

$\vec{AB}^{\perp} = \langle 3, -1\rangle$

$\vec{PA} = \langle 0, 5\rangle - \langle -2, 3\rangle = \langle 2, 2\rangle$

$$\vec{PH} = \left| \frac{\vec{PA} \cdot \vec{AB}^{\perp}}{|\vec{AB}^{\perp}|} \right| = \left| \frac{\langle 2,2\rangle \cdot \langle 3,-1\rangle}{|\langle 3,-1\rangle|} \right|$$

$$= \left| \frac{2 \cdot 3 + 2 \cdot (-1)}{\sqrt{3^2 + (-1)^2}} \right| = \frac{4}{\sqrt{10}}$$