Section 1.3: Vector functions

Parametric equations:

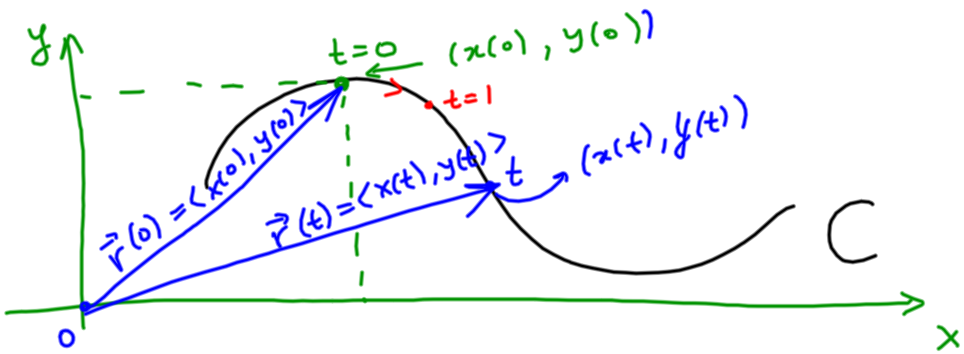
$$x = x(t), \quad y = y(t)$$

where the variable t is called a **parameter.** Each value of the parameter t defines a point that we can plot. As t varies over its domain we get a collection of points $(x, y) = (x(t), y(t))$ on the plane which is called the **parametric curve.**

Each parametric curve can be represented as the **vector function**:

$$\overrightarrow{r(t)} = \langle x(t), y(t) \rangle \, .$$

Note that Parametric curves have a **direction of motion** given by increasing of parameter t. So, when sketching parametric curves we also include arrows that show the direction of motion.



EXAMPLE 1. *Examine the parametric curve* $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 3\pi/2.$

parameter domain

$\vec{r}(t) = \langle \cos t, \ \sin t \rangle$

t	0	$\pi/4$	$\pi/2$	π	$\ldots$	$3\pi/2$
(x,y)	$(1,0)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$(0,1)$	$(-1,0)$	$\ldots$	$(0,-1)$

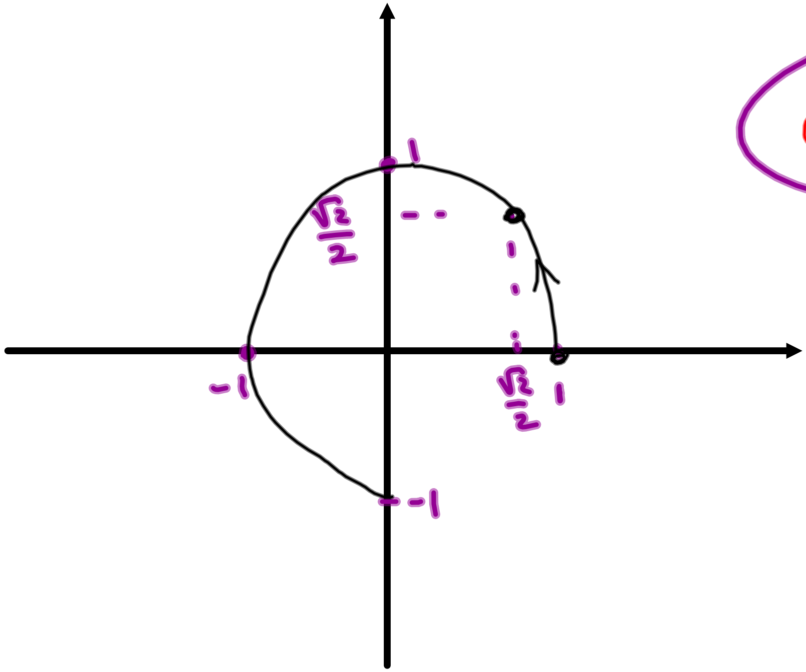
$(\cos t, \sin t)$

Eliminate parameter

$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

Cartesian form



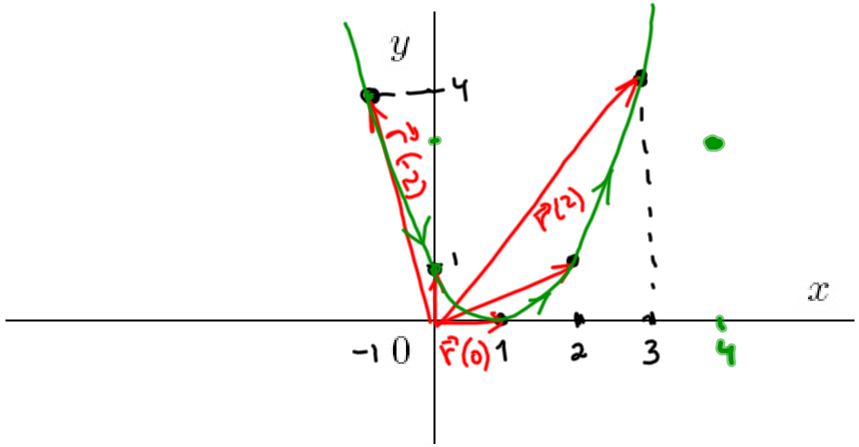
EXAMPLE 2. *Given* $\mathbf{r}(t) = \langle t+1, t^2 \rangle$. or $x(t) = t+1,\ y(t) = t^2$

(a) *Does the point* $(4,3)$ *belong to the graph of* $\mathbf{r}(t)$*?*

$\vec{r}(t) = \langle 4, 3 \rangle \Rightarrow \begin{cases} t+1 = 4 & \Rightarrow t = 3 \\ t^2 = 3 & \Rightarrow t = \pm\sqrt{3} \end{cases} \Rightarrow$ NO

(b) *Sketch the graph of* $\mathbf{r}(t)$.

t	$\mathbf{r}(t)$
-2	$\langle -1, 4 \rangle$
-1	$\langle 0, 1 \rangle$
0	$\langle 1, 0 \rangle$
1	$\langle 2, 1 \rangle$
2	$\langle 3, 4 \rangle$



(c) *Find the Cartesian equation of* $\mathbf{r}(t)$ *eliminating the parameter.*

$x = t+1 \Rightarrow t = x-1$

$y = t^2 \longrightarrow \boxed{y = (x-1)^2}$ parabola

EXAMPLE 3. *Find the Cartesian equation for* $\mathbf{r}(t) = \cos t\mathbf{i} + \cos(2t)\mathbf{j}$

Eliminate parameter

$$\begin{cases} x = \cos t \\ y = \cos 2t \end{cases}$$

$\cos 2t = 2\cos^2 t - 1$

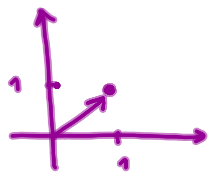
$\cdot y = 2x^2 - 1$ parabola

EXAMPLE 4. *An object is moving in the xy-plane and its position after t seconds is given by* $\mathbf{r}(t) = \langle 1+t^2, 1+3t \rangle$.

$x = 1 + t^2 \quad , \quad y = 1 + 3t$

(a) *Find the position of the object at time $t = 0$.*

$\vec{r}(0) = \langle 1 + 0^2 , 1 + 3\cdot 0 \rangle = \langle 1, 1 \rangle$



(b) *At what time does the object reach the point $(10, 10)$.*

$\vec{r}(t) = \langle 10 , 10 \rangle$

$$\begin{cases} 1 + t^2 = 10 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3 \\ 1 + 3t = 10 \Rightarrow 3t = 9 \Rightarrow t = 3 \end{cases} \Rightarrow t = 3$$

(c) *Does the object pass through the point $(20, 20)$?*

$1 + t^2 = 20 \Rightarrow t^2 = 19 \Rightarrow t = \pm\sqrt{19}$

$1 + 3t = 20 \Rightarrow 3t = 19 \Rightarrow t = \frac{19}{3}$

NO

(d) *Find an equation in x and y whose graph is the path of the object.*

Eliminate parameter t to find Cartesian form

$x = 1 + t^2$

$y = 1 + 3t \Rightarrow t = \frac{y-1}{3}$

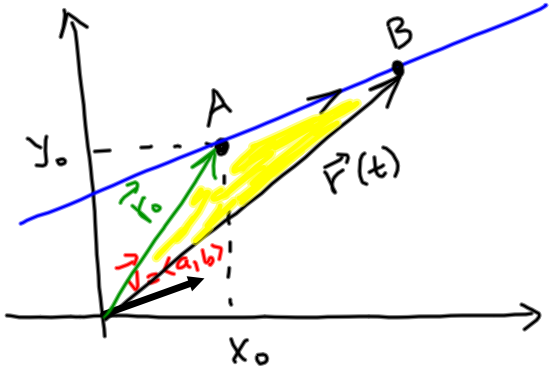
$$x = 1 + \left(\frac{y-1}{3}\right)^2 \Rightarrow \boxed{x = 1 + \frac{(y-1)^2}{9}}$$

A **Vector equation of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b\rangle$ is given by

$\vec{r}_0 = \langle x_0, y_0 \rangle$

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v},$$

where $\mathbf{r_0} = \langle x_0, y_0\rangle$.



$$\vec{r}(t) = \vec{r}_0 + \vec{AB} = \vec{r}_0 + t\vec{V}$$

$$\vec{AB} \parallel \vec{V}$$

$$\vec{AB} = t\vec{V}$$

The vector equation of the line can be rewritten in parametric form. Namely, we have

point on the line

components of $\vec{v}$

$$\begin{aligned} \langle x(t), y(t)\rangle &= \mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \\ &= \langle x_0, y_0\rangle + t\langle a, b\rangle = \langle x_0, y_0\rangle + \langle ta, tb\rangle = \\ &= \langle x_0 + ta, y_0 + tb\rangle . \end{aligned}$$

$$x = x_0 + ta$$
$$y = y_0 + tb$$

This immediately yields that the **parametric equations of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b\rangle$ are

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt.$$

EXAMPLE 5. *Find parametric equations of the line*

(a) *passing through the point* $\underset{x_0, y_0}{(1,0)}$ *and parallel to the vector* $\mathbf{i} - 4\mathbf{j}$; $\vec{v} = \langle 1, -4 \rangle$

$$x = 1 + t \cdot 1$$
$$y = 0 + t \cdot (-4)$$

or

$$\boxed{\begin{aligned} x &= 1 + t \\ y &= -4t \end{aligned}}$$

Remark: in vector form: $\vec{r}(t) = \langle 1+t, -4t \rangle$

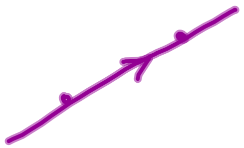
(b) *passing through the point* $(-4, 5)$ *with slope* $\sqrt{3}$; $\Rightarrow \tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$$\vec{v} = \langle \cos \alpha, \sin \alpha \rangle = \langle \cos \tfrac{\pi}{3}, \sin \tfrac{\pi}{3} \rangle = \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$

$$\boxed{\begin{aligned} x &= -4 + \tfrac{1}{2} t \\ y &= 5 + \tfrac{\sqrt{3}}{2} t \end{aligned}}$$

(c) *passing through the points* $(7, 2)$ *and* $(3, 2)$.

$$\vec{v} = \langle 3, 2 \rangle - \langle 7, 2 \rangle = \langle -4, 0 \rangle$$



$$x = 7 + t \cdot (-4)$$
$$y = 2 + t \cdot 0$$

or

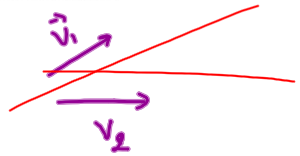
$$\boxed{\begin{aligned} x &= 7 - 4t \\ y &= 2 \end{aligned}}$$

EXAMPLE 6. *Determine whether the lines* $\mathbf{r}(t) = \langle 1+t, 1-3t\rangle$, $\mathbf{R}(s) = \langle 1+3s, 12+s\rangle$ *are parallel, orthogonal or neither. If they are not parallel, find the intersection point.*

L1 L2

L1 : $\vec{V}_1 = \langle 1, -3\rangle$

L2 : $\vec{V}_2 = \langle 3, 1\rangle = \vec{V}_1^{\perp}$



OR

$\vec{V}_1 \cdot \vec{V}_2 = 1\cdot 3 + (-3)\cdot 1 = 0 \rightarrow$ L1 $\perp$ L2 orthogonal

$\vec{r}(t) = \vec{R}(s)$

$\langle 1+t, 1-3t\rangle = \langle 1+3s, 12+s\rangle$

$1+t = 1+3s \quad (\times 3)$

$1-3t = 12+s$

$+ \quad 3+3t = 3+9s$

$4 = 15 + 10s \Rightarrow 10s = -11 \Rightarrow s = -\frac{11}{10}$

$t = 3s = 3\cdot\left(-\frac{11}{10}\right) = -\frac{33}{10}$

$\vec{r}(t) = \langle 1 - \frac{33}{10}, 1 - 3\cdot\left(-\frac{33}{10}\right)\rangle$

$= \langle \frac{10-33}{10}, \frac{10+99}{10}\rangle$

$= \langle -\frac{23}{10}, \frac{109}{10}\rangle$

$\vec{R}\left(-\frac{11}{10}\right) = \langle 1+3\cdot\left(-\frac{11}{10}\right), 12+\left(-\frac{11}{10}\right)\rangle$

$= \langle 1 - \frac{33}{10}, 12 - \frac{11}{10}\rangle$

$= \langle \frac{10-33}{10}, \frac{120-11}{10}\rangle$

$= \langle -\frac{23}{10}, \frac{109}{10}\rangle$

Answer: orthogonal

intersection point is $\left(-\frac{23}{10}, \frac{109}{10}\right)$