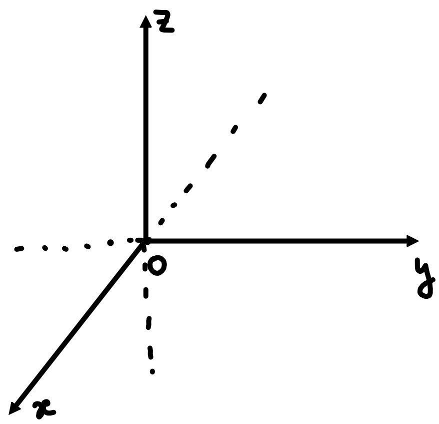
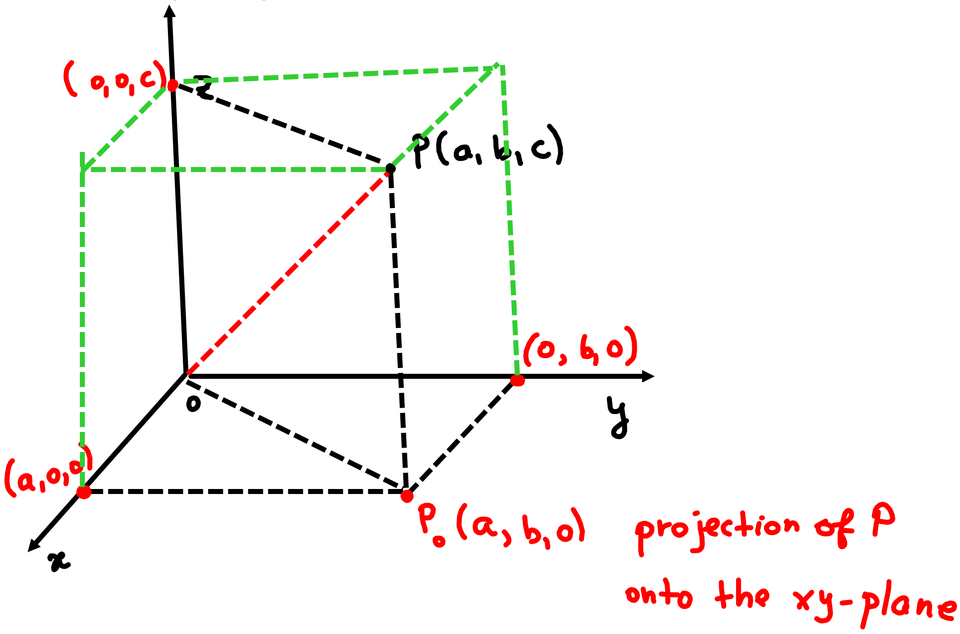
11.1: Three-dimensional Coordinate System

$\mathbb{R}^3$

The three-dimensional coordinate system consists of the **origin** O and the **coordinate axes**: x-axis, y-axis, z-axis.The coordinate axes determine 3 **coordinate planes**: the xy-plane, the xz-plane and yz-plane. The coordinate planes divide space into 8 parts, called octants.



Representation of point $P(a, b, c)$ and its projections on the coordinate planes:



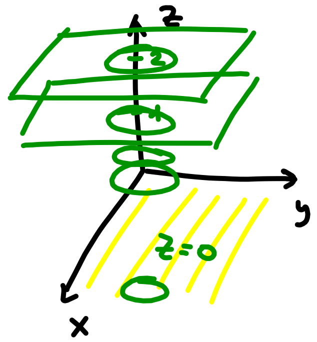
EXAMPLE 1. *Describe in words the regions of $\mathbb{R}^3$ represented by the following equation:*

(a) $\boxed{z = 0}$ is equation of the xy- plane

$(x, y, 0)$

$z = 1$ $(x, y, 1)$

$z = c$ (x, y, c)



(b) $\boxed{y = 0}$ is equation of the xz-plane

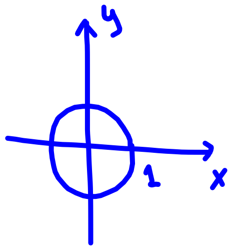
(c) $\boxed{x = 0}$ is equation of the yz-plane

Note that in $\mathbb{R}^2$ the graph of the equation involving x and y is a curve. In $\mathbb{R}^3$ an equation in x, y, z represents a **surface**.(It does not mean that we cant graph curves in $\mathbb{R}^3$.)

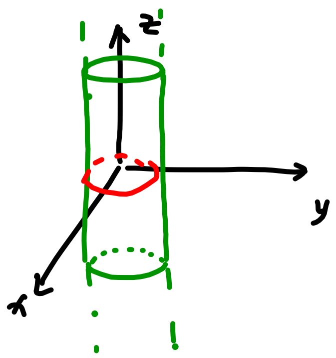
EXAMPLE 2. *Sketch the graph of* $x^2 + y^2 - 1 = 0$ *in* $\mathbb{R}^2, \mathbb{R}^3$.

$x^2 + y^2 = 1$

$\mathbb{R}^2$



$\mathbb{R}^3$ circular cylinder

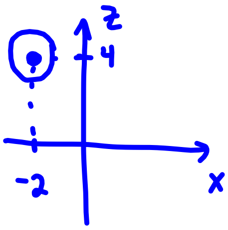


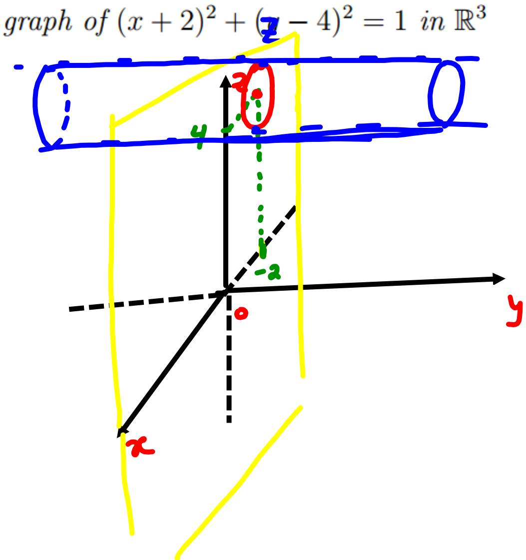
An equation that contains only two of the variables x, y, z represents a **cylindrical surface** *in* $\mathbb{R}^3$. *How to graph a cylindrical surface:*

1. *graph the equation in the coordinate plane of the two variables that appear in the given equation;*

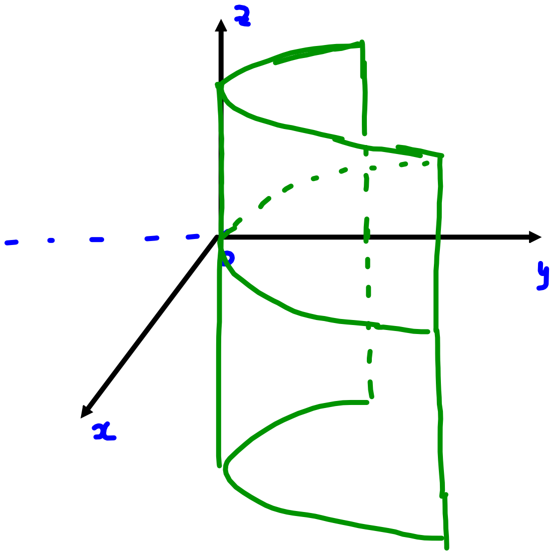
2. *translate that graph parallel to the axis of the missing variable.*

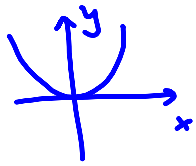
EXAMPLE 3. *Sketch the graph of* $(x+2)^2 + (y-4)^2 = 1$ *in* $\mathbb{R}^3$





EXAMPLE 4. *Sketch the graph of $y = x^2$ in $\mathbb{R}^3$*



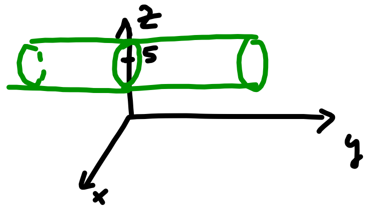


parabolic cylinder

EXAMPLE 5. *Let S be the graph of $x^2 + z^2 - 10z + 21 = 0$ in $\mathbb{R}^3$.*

(a) *Describe S.*

$(a \pm b)^2 = a^2 \pm 2ab + b^2$



$x^2 + z^2 - 10z + 21 = 0$

$x^2 + \underbrace{z^2 - 2\cdot z \cdot 5 + 5^2} - 5^2 + 21 = 0$

$x^2 + (z-5)^2 - 25 + 21 \Rightarrow \boxed{x^2 + (z-5)^2 = 4}$

circular cylinder parallel to the y-axis with radius 2.

(b) *The intersection of S with the xz- plane is* circle with radius 2 centered at $(0,0,5)$ in the xz-plane

$$\begin{cases} x^2 + z^2 - 10z + 21 = 0 \\ y = 0 \end{cases} \Rightarrow$$

(c) *The intersection of S with the yz- plane is* two horizontal lines through $(0,0,7)$ & $(0,0,3)$

$$\begin{cases} x^2 + (z-5)^2 = 4 \\ x = 0 \end{cases} \Rightarrow (z-5)^2 = 4 \Rightarrow z - 5 = \pm 2$$

$\nearrow z - 5 = 2 \Rightarrow z = 7$

$\searrow z - 5 = -2 \Rightarrow z = 3$

(d) *The intersection of S with the xy- plane is* empty

$$\begin{cases} x^2 + (z-5)^2 = 4 \\ z = 0 \end{cases} \Rightarrow x^2 + (0-5)^2 = 4$$

$x^2 = 4 - 25$

$x^2 = -21$ impossible

$x^2+y^2+z^2=1$ sphere with radius 1 centered at origin

$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}=1$

$d\big((x,y,z),(0,0,0)\big)=1$

Spheres

- **Distance formula in** $\mathbb{R}^3$: The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

EXAMPLE 6. *Show that the equation* $x^2+y^2+z^2+2x-4y+8z+17=0$ *represents a sphere, and find its center and radius.*

Complete squares $(a\pm b)^2 = a^2 \pm 2ab + b^2$

$\rightarrow x^2+2x+1$

$+\, y^2-4y+4$

$+\, z^2+8z+16$

$= -17+1+4+16$

$(x+1)^2+(y-2)^2+(z+4)^2=4$ sphere centered at $(-1,2,-4)$ with radius 2.

$d\big((x,y,z),(-1,2,-4)\big)=2$

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

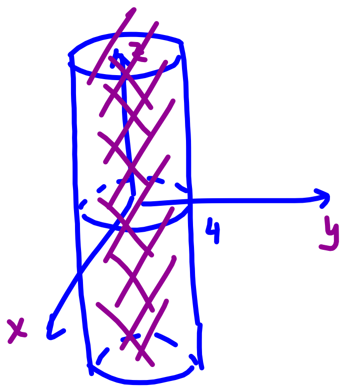
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k$$

- If $k > 0$ then the graph of this equation is Spere centered at (a, b, c) and radius $\sqrt{k}$
- If $k = 0$, then the graph is point (a, b, c)
- If $k < 0$ then no points

Regions in $\mathbb{R}^3$

EXAMPLE 7. *Describe the set of all points in $\mathbb{R}^3$ whose coordinates satisfy the following inequality:* $x^2 + y^2 < 16$

$x^2+y^2=16$ cylinder



Test point $(0,0,0)$

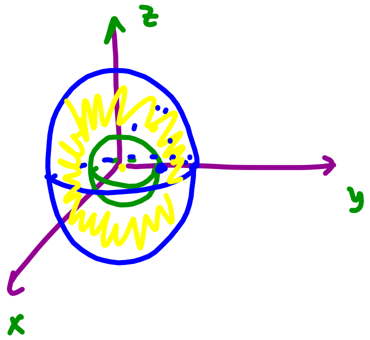
$0 < 16$ (T)

All points inside the cylinder $x^2+y^2=16$

not including the boundary surface

EXAMPLE 8. *Describe the following region:* $\{(x, y, z) | 9 \leq x^2 + y^2 + z^2 \leq 16\}$

$$\left.\begin{array}{l} x^2+y^2+z^2=9 \\ x^2+y^2+z^2=16 \end{array}\right\} \text{ spheres centered at } (0,0,0) \text{ with radii 3 and 4}$$



$$3 < \sqrt{x^2+y^2+z^2} < 4$$

All points between two concentric spheres centered at origin with radii 3 and 4 including these spheres.