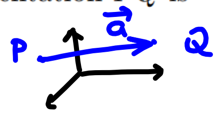
# 11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \vec{a} = \overrightarrow{OA}$

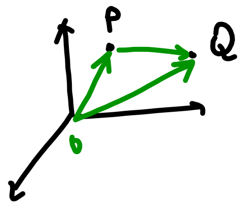
where $A(a_1, a_2, a_3)$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

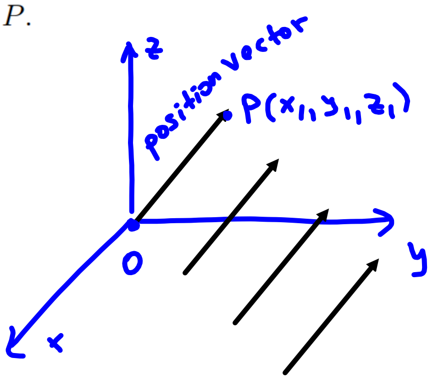
$$\mathbf{a} = \langle \underbrace{x_2 - x_1, y_2 - y_1, z_2 - z_1}_{Q-P} \rangle = \overrightarrow{PQ}$$




The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point $P$.




$$\overrightarrow{PQ} = \overrightarrow{OQ} \underbrace{- \overrightarrow{OP}}$$
$$= \overrightarrow{OQ} + \overrightarrow{PO}$$
$$= \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{PQ}$$



EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1,2,3)$ *and terminal point* $B(3,2,-1)$. *What is the position vector of the point* $A$?

$$\overrightarrow{AB} = \langle 3-1,\ 2-2,\ -1-3 \rangle = \langle 2, 0, -4 \rangle$$

position vector of $A(1,2,3)$:

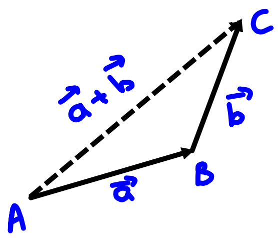
$$\overrightarrow{OA} = \langle 1, 2, 3 \rangle$$

**Vector Arithmetic:** Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.

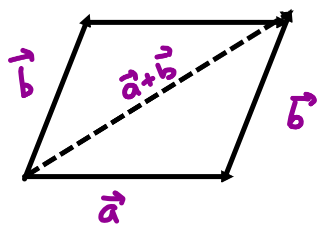
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW

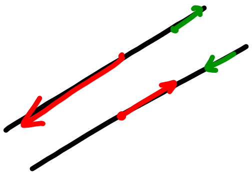



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

PARALLELOGRAM LAW



Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:



$$\mathbf{a} \| \mathbf{b} \iff \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

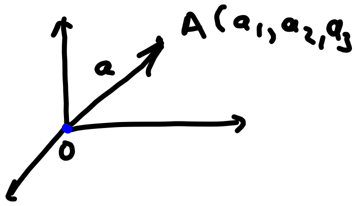
$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \underbrace{\langle b_1, b_2, b_3 \rangle} = \alpha \vec{a} = \underbrace{\langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle}$

$$\left.\begin{aligned} b_1 &= \alpha a_1 \\ b_2 &= \alpha a_2 \\ b_3 &= \alpha a_3 \end{aligned}\right\} \Rightarrow \alpha = \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$$

$\langle 3, 4, 8 \rangle \| \langle -9, -12, -24 \rangle = -3 \langle 3, 4, 8 \rangle$

**The magnitude or length** of $a = \langle a_1, a_2, a_3 \rangle$:

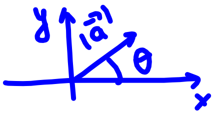



$$|\vec{OA}| = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector in the same direction as a:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vector* $\mathbf{a}$ *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector $\mathbf{a}$ can be determined by its length and the angle from the positive $x$-axis:




$$\vec{a} = |\vec{a}| \underbrace{\langle \cos\theta, \sin\theta \rangle}_{\hat{a}} = |\vec{a}| \cdot \hat{a}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

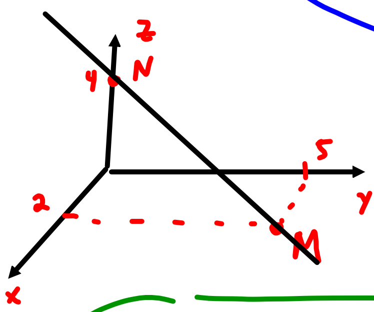
$$\boxed{\mathbf{a} = |\mathbf{a}|\, \hat{\mathbf{a}},}$$

i.e. $\mathbf{a}$ is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2,5,0)$ *and* $N(0,0,4)$.

$$|\vec{a}| = \sqrt{5}$$

$$\vec{a} \parallel \vec{MN}$$




$$\vec{a} = \pm|\vec{a}|\cdot\widehat{MN} = \pm|\vec{a}|\cdot\frac{\vec{MN}}{|\vec{MN}|}$$

$$\vec{a} = \pm\cancel{\sqrt{5}}\,\frac{\langle -2,-5,4\rangle}{3\cancel{\sqrt{5}}} = \pm\left\langle -\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}\right\rangle$$

$$\vec{MN} = \langle 0-2, 0-5, 4-0\rangle = \langle -2,-5,4\rangle$$

$$|\vec{MN}| = \sqrt{(-2)^2+(-5)^2+4^2} = \sqrt{45} = \sqrt{9\cdot 5} = 3\sqrt{5}$$

**Standard Basis Vectors:**

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\langle -1, 2, 3 \rangle = -\hat{i} + 2\hat{j} + 3\hat{k}$$

EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3\rangle$ *and* $\mathbf{b} = \langle 3, 1, 2\rangle$. *Find*

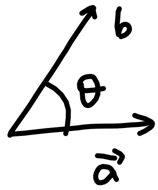
**(a)** $|\mathbf{b} - \mathbf{a}| = |\langle 3-1, 1-0, 2-(-3)\rangle| = |\langle 2, 1, 5\rangle|$

$$= \sqrt{2^2+1^2+5^2} = \sqrt{30}$$

**(b)** *a unit vector that has the same direction as* $\mathbf{b}$.

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\langle 3,1,2\rangle}{\sqrt{3^2+1^2+2^2}} = \frac{\langle 3,1,2\rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$$

**Dot Product** of two nonzero vectors **a** and **b** is ~~the~~ a NUMBER:




$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta,$$

where $\theta$ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

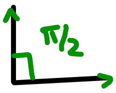
If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a}\cdot\mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If $\theta$ is the *angle* between two nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$$




DEFINITION 5. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

EXAMPLE 6. *For two nonzero vectors* **a** *and* **b** *show that*

(a)

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

$\Rightarrow$ Show $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$\Downarrow$

$\theta = \frac{\pi}{2}$

$\Downarrow$ $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos\frac{\pi}{2}}_{=0}$

(b)

$$\boxed{|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}}$$

$\sqrt{a \cdot a} = \sqrt{|\vec{a}| \cdot |\vec{a}| \cos 0} = \sqrt{|\vec{a}|^2} = |\vec{a}|$

$\Leftarrow$ Show $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\Downarrow$

$\underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cdot \cos\theta = 0$

$\Downarrow$

$\cos\theta = 0 \Rightarrow \theta = \pi/2$

EXAMPLE 7. *For what value(s) of $c$ are the vectors $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ orthogonal?*

$\langle c, 2, 1\rangle \perp \langle 4, 3, c\rangle$

if and only if

$\langle c, 2, 1\rangle \cdot \langle 4, 3, c\rangle = 0$

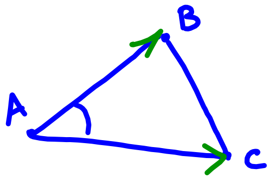
$c \cdot 4 + 2 \cdot 3 + 1 \cdot c = 0$

$4c + 6 + c = 0$

$5c = -6$

$\boxed{c = -\frac{6}{5}}$

EXAMPLE 8. *The points* $A(6,-1,0)$, $B(-3,1,2)$, $C(2,4,5)$ *form a triangle. Find angle at* $A$.




$$\cos \measuredangle A = \cos \measuredangle \vec{AB}, \vec{AC} = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}$$

$$\vec{AB} = \langle -3-6,\ 1-(-1),\ 2-0 \rangle = \langle -9, 2, 2 \rangle$$

$$\vec{AC} = \langle 2-6,\ 4-(-1),\ 5-0 \rangle = \langle -4, 5, 5 \rangle$$

$$|\vec{AB}| = \sqrt{81+4+4} = \sqrt{89}$$

$$|\vec{AC}| = \sqrt{16+25+25} = \sqrt{66}$$

$$\cos \measuredangle A = \frac{\langle -9, 2, 2 \rangle \cdot \langle -4, 5, 5 \rangle}{\sqrt{89} \cdot \sqrt{66}} = \frac{36+10+10}{\sqrt{89 \cdot 66}}$$

$$= \frac{56}{\sqrt{89 \cdot 66}}$$

$\measuredangle A = ?$ (calculator)