

11.3: Cross product

REVIEW: Determinant of 2×2 and 3×3 matrices.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

or copy the first two columns onto the end and then multiply along each diagonal and add those that move from left to right and subtract those that move from right to left:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

• **THE CROSS PRODUCT IN COMPONENT FORM:**

Given $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then the cross product is :

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

REMARK 1. The cross product requires both of the vectors to be three dimensional vectors.

REMARK 2. The result of a dot product is a number and the result of a cross product is a VECTOR!!!

To remember the cross product component formula use the fact that the cross product can be represented as the determinant of order 3:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{i} (a_2b_3 - a_3b_2) - \hat{j} (a_1b_3 - b_1a_3) \\ &\quad + \hat{k} (a_1b_2 - a_2b_1) \end{aligned}$$

EXAMPLE 3. If $\mathbf{a} = \langle -2, 1, 1 \rangle$, $\mathbf{b} = \langle 3, 5, 0 \rangle$, and $\mathbf{c} = \langle -4, 2, 2 \rangle$ compute each of the following:

$$\begin{aligned} \text{(a) } \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 3 & 5 & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} -2 & 1 \\ 3 & 5 \end{vmatrix} \\ &= \hat{i} (1 \cdot 0 - 1 \cdot 5) - \hat{j} (-2 \cdot 0 - 1 \cdot 3) + \hat{k} (-2 \cdot 5 - 1 \cdot 3) \\ &= \langle -5, 3, -13 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{b} \times \mathbf{a} = -\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \langle 5, -3, 13 \rangle$$

$$\text{(c) } \mathbf{a} \times (5\mathbf{b}) = 5(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \langle -25, 15, -65 \rangle$$

$$\text{(d) } \mathbf{a} \times \mathbf{a} = \vec{\mathbf{0}}$$

$$\text{(e) } \mathbf{a} \times \mathbf{c} = \mathbf{a} \times (2\vec{\mathbf{a}}) = 2(\vec{\mathbf{a}} \times \vec{\mathbf{a}}) = \vec{\mathbf{0}}$$

Properties:

$$\# \mathbf{a} \times \mathbf{a} = \vec{\mathbf{0}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$* (\alpha \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha \mathbf{b}) = \alpha(\mathbf{a} \times \mathbf{b}), \quad \alpha \in \mathbb{R}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

EXAMPLE 4. Show that if \mathbf{a} and \mathbf{b} are parallel then $\mathbf{a} \times \mathbf{b} = \vec{\mathbf{0}}$.

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there exists a real scalar c such that

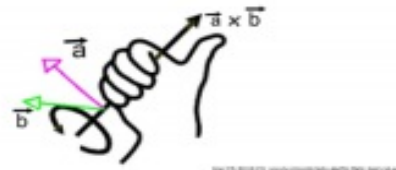
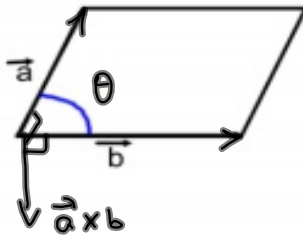
$$\vec{\mathbf{b}} = c \vec{\mathbf{a}}$$

Then
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times (c \vec{\mathbf{a}}) \stackrel{*}{=} c (\vec{\mathbf{a}} \times \vec{\mathbf{a}}) \stackrel{\#}{=} c \cdot \vec{\mathbf{0}} = \vec{\mathbf{0}}$$

• **GEOMETRIC INTERPRETATION OF THE CROSS PRODUCT:**

Let θ be the angle between the two nonzero vectors \mathbf{a} and \mathbf{b} , $0 \leq \theta \leq \pi$. Then

1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin \theta =$ *the area of the parallelogram determined by \mathbf{a} and \mathbf{b}* ;
2. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} ;
3. the direction of $\mathbf{a} \times \mathbf{b}$ is determined by “right hand” rule: *if the fingers of your right hand curl through the angle θ from \mathbf{a} to \mathbf{b} , then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.*



For two nonzero vectors \vec{a} and \vec{b}

FACT: $a \parallel b \Leftrightarrow a \times b = \vec{0}$.

By example 4, $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} \times \vec{b} = \vec{0}$.

Conversely, let $\vec{a} \times \vec{b} = \vec{0} \Rightarrow |\vec{a} \times \vec{b}| = |\vec{0}| = 0$

$$\overset{\parallel}{|\vec{a}| \cdot |\vec{b}| \cdot \sin \theta = 0}$$

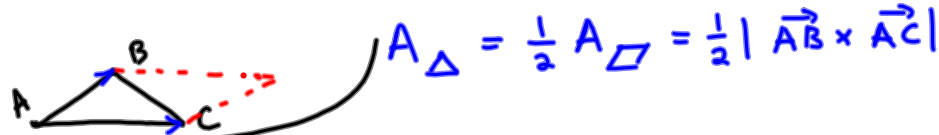
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$$\Downarrow$$
$$\sin \theta = 0$$

$$\Downarrow$$
$$\theta = 0 \Rightarrow \vec{a} \parallel \vec{b}$$

EXAMPLE 5. Given the points $A(1, 0, 0)$, $B(1, 1, 1)$ and $C(2, -1, 3)$. Find

(a) the area of the triangle determined by these points.



$$A_{\Delta} = \frac{1}{2} A_{\square} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

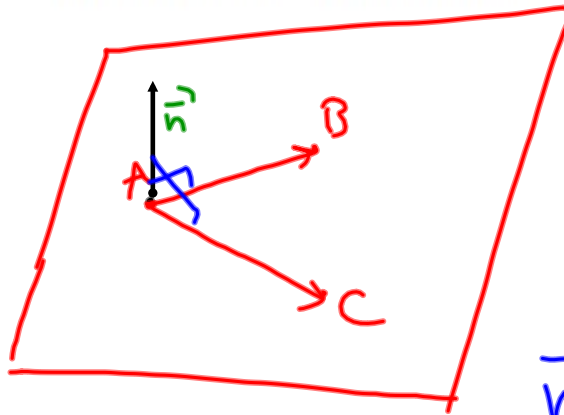
$$\vec{AB} = \langle 0, 1, 1 \rangle \quad \vec{AC} = \langle 1, -1, 3 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \hat{i}(3+1) - \hat{j}(0-1) + \hat{k}(0-1) \\ = \langle 4, 1, -1 \rangle$$

$$|\vec{AB} \times \vec{AC}| = |\langle 4, 1, -1 \rangle| = \sqrt{16+1+1} = \sqrt{18} = \sqrt{2 \cdot 9} = 3\sqrt{2}$$

$$\rightarrow A_{\Delta} = \frac{1}{2} \cdot 3\sqrt{2} = \frac{3\sqrt{2}}{2}$$

(b) Find a unit vector \hat{n} orthogonal to the plane that contains the points A, B, C.



$$\left. \begin{array}{l} \vec{n} \perp \vec{AB} \\ \vec{n} \perp \vec{AC} \end{array} \right\} \Rightarrow \vec{n} \parallel \vec{AB} \times \vec{AC}$$

$$\vec{n} = \langle 4, 1, -1 \rangle$$

$$|\vec{n}| = 3\sqrt{2}$$

$$\hat{n} = \pm \frac{\vec{n}}{|\vec{n}|} = \pm \frac{\langle 4, 1, -1 \rangle}{3\sqrt{2}}$$

MIXED

- SCALAR TRIPLE PRODUCT of the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \rightarrow \text{a number}$$

$\underbrace{\hspace{1.5cm}}_{\text{a vector}}$

Note that the scalar triple product is a NUMBER.

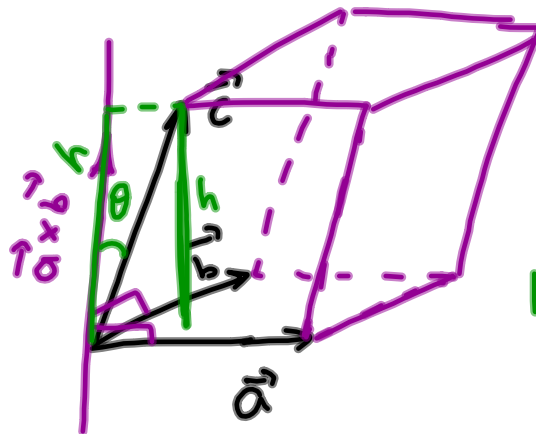
FACTS:

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

2. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3. $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \text{the volume of the parallelepiped determined by } \mathbf{a}, \mathbf{b}, \mathbf{c}.$

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$



$$\begin{aligned} V &= (\text{base}) \cdot (\text{height}) \\ &= |\mathbf{a} \times \mathbf{b}| \cdot |\mathbf{c}| \cos \theta \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

$$h = |\mathbf{c}| \cdot \cos \theta$$

EXAMPLE 6. Determine if the vectors

$$\mathbf{a} = \langle 0, -9, 18 \rangle, \quad \mathbf{b} = \langle 1, 4, -7 \rangle, \quad \mathbf{c} = \langle 2, -1, 4 \rangle$$

are coplanar.²

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{vmatrix} 0 & -9 & 18 \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix} = -(-9) \begin{vmatrix} 1 & -7 \\ 2 & 4 \end{vmatrix} + 18 \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} \\ &= 9(4 + 14) + 18(-1 - 8) \\ &= 9 \cdot 18 - 18 \cdot 9 = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are} \\ &\quad \text{coplanar.} \end{aligned}$$