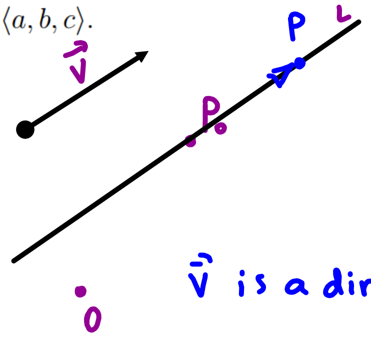
11.4: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$.



$P(x,y,z)$

$\vec{v} \parallel L \parallel \overrightarrow{P_0P} \Rightarrow \overrightarrow{P_0P} = t\vec{v}$

for some real t

$\langle x-x_0, y-y_0, z-z_0 \rangle = \langle ta, tb, tc \rangle$

$x - x_0 = ta$

$y - y_0 = tb$

$z - z_0 = tc$

$\vec{v}$ is a direction vector.

Parametric equations of the line:

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

$$\underbrace{\begin{aligned} x &= -1 + 3t \\ y &= 3 - t \\ z &= 7t \end{aligned}}$$

param. equation of the line through the point $(-1, 3, 0)$ and parallel to $\vec{v}\langle 3, -1, 7 \rangle$

$$\begin{aligned} x &= -1 + 6t \\ y &= 3 - 2t \\ z &= 14t \end{aligned}$$

Parametric equations of the line:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

EXAMPLE 1. *Find parametric equations of the line*

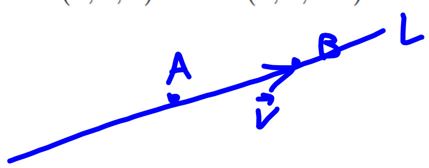
(a) *passing through the point* $(3, -4, 1)$ *and parallel to* $\mathbf{v} = \langle 7, 0, -1 \rangle$

$$\begin{aligned} x &= 3 + 7t \\ y &= -4 + 0t \\ z &= 1 + (-1)t \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= 3+7t \\ y &= -4 \\ z &= 1-t \end{aligned}}$$

(b) *passing through the origin* $(0,0,0)$ *and parallel to* $\mathbf{v} = \langle 5, 5, 5 \rangle$

$$\begin{aligned} x &= 5t \\ y &= 5t \\ z &= 5t \end{aligned}$$

EXAMPLE 2. *Consider the line L that passes through the points $A(1,1,1)$ and $B(2,3,-2)$. Find points at that L intersects the yz-plane.* $(x=0)$



Find parametric equations of L.

$\vec{v} = \overrightarrow{AB} = \langle 1, 2, -3 \rangle$

$$L: \begin{cases} x = 1 + t \\ y = 1 + 2t \\ z = 1 - 3t \end{cases}$$

$x = 0$

$1 + t = 0 \Rightarrow t = -1$

$y = 1 + 2\cdot(-1) = -1$

$z = 1 - 3\cdot(-1) = 4$

$$L \cap \{x=0\} = \boxed{(0, -1, 4)}$$

$x = x_0 + at$ $\Rightarrow$ $t = \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$

$y = y_0 + bt$

$z = z_0 + ct$

Symmetric equations of the line: If $abc \neq 0$ then

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

EXAMPLE 3. *Find symmetric equations of lines from Example 1.*

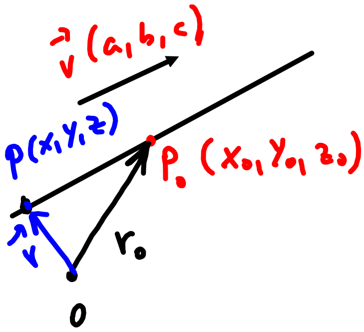
(a) $\frac{x-3}{7} = \frac{z-1}{-1}$, $y = -4$

(b) $\frac{x-0}{5} = \frac{y-0}{5} = \frac{z-0}{5}$

Param. equations

$$\underbrace{\begin{matrix} x \\ y \\ z \end{matrix}}_{\vec{r}} \begin{matrix} = \\ = \\ = \end{matrix} \underbrace{\begin{matrix} x_0 \\ y_0 \\ z_0 \end{matrix}}_{\vec{r}_0} \begin{matrix} + \\ + \\ + \end{matrix} \underbrace{\begin{matrix} a \\ b \\ c \end{matrix}}_{\vec{v}} \begin{matrix} t \\ t \\ t \end{matrix}$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$



Vector equation of the line:

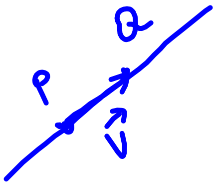
$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c \rangle$ is some vector which is parallel to the line, t is a parameter, $-\infty < t < \infty$.

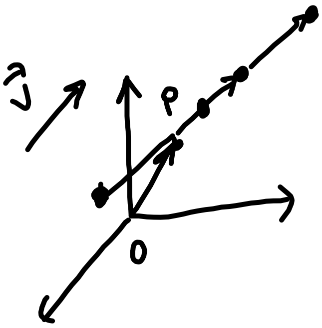
EXAMPLE 4. *Find vector equation of the line that passes through the points* $P(1,1,-4)$ *and* $Q(0,3,-4)$.

Direction vector

$$\vec{v} = \vec{PQ} = \langle 0-1, 3-1, -4-(-4)\rangle$$
$$= \langle -1, 2, 0\rangle$$



$$\boxed{\vec{r}(t) = \underbrace{\langle 1,1,-4\rangle}_{\vec{OP}} + t\langle -1,2,0\rangle}$$



$t=1$ $\quad \vec{OP} + \vec{v}$

$t=2$ $\quad \vec{OP} + 2\vec{v}$

$t=\frac{1}{2}$ $\quad \vec{OP} + \frac{1}{2}\vec{v}$

EXAMPLE 5. *Determine whether the lines*

$\vec{v}_1 = \langle 1, 3, -1 \rangle$

$$L_1: \quad \frac{x-1}{1} = \frac{y+2}{3} = \frac{z-4}{-1} \qquad P_1(1,-2,4)$$

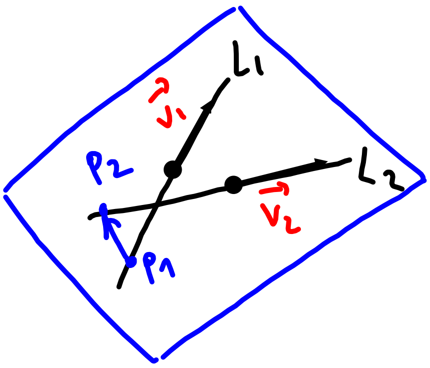
and

$\vec{v}_2 = \langle 2, 1, 4 \rangle$

$$L_2: \quad x = 2t, \quad y = 3+t, \quad z = -3+4t \qquad P_2(0,3,-3)$$

are ~~*parallel*~~, *skew, or intersecting.*

$\vec{v}_1 \nparallel \vec{v}_2 \qquad \frac{1}{2} \neq \frac{3}{1} \neq \frac{-1}{4}$



If L_1 and L_2 are intersecting then $\vec{v}_1$, $\vec{v}_2$ and $\overrightarrow{P_1P_2} = \langle -1, 5, -7 \rangle$ must be coplanar, i.e.

$$(\vec{v}_1 \times \vec{v}_2) \cdot \overrightarrow{P_1P_2} = 0$$

Otherwise, L_1 and L_2 are skew.

$$(\vec{v}_1 \times \vec{v}_2) \cdot \overrightarrow{P_1P_2} = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ -1 & 5 & -7 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 4 \\ 5 & -7 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ -1 & -7 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix}$$

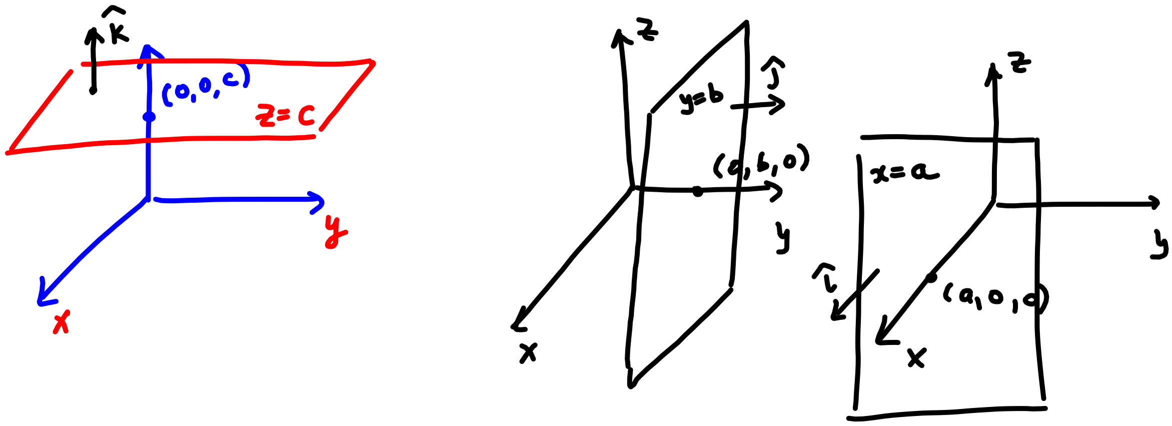
$$= -7 - 20 - 3(-14+4) - (10+1)$$

$$= -27 + 30 - 11 \neq 0$$

$\Rightarrow L_1$ and L_2 are skew.

Planes

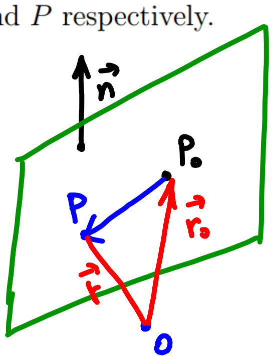
Planes parallel to the coordinate planes:



Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.



$\overrightarrow{P_0P} \perp \vec{n}$

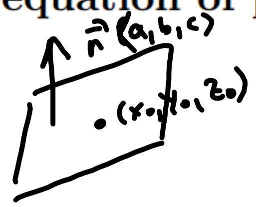
$\overrightarrow{P_0P} \cdot \vec{n} = 0$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Vector equation of the plane: $\quad \mathbf{n} \cdot \overbrace{(\mathbf{r} - \mathbf{r_0})}^{\overrightarrow{P_0P}} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}.$

Scalar equation of plane:



$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 6. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

P_0 $\vec{n}$

$$2(x-1) + 3(y-2) + 4(z-1) = 0$$

OR

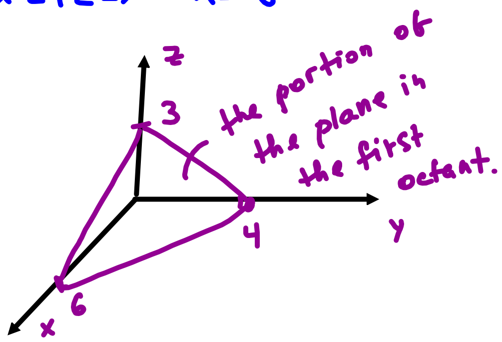
$$2x + 3y + 4z = 12$$

the x-intercept : $y=0$, $z=0$ $\Rightarrow$ $2x = 12 \Rightarrow$ $x = 6$

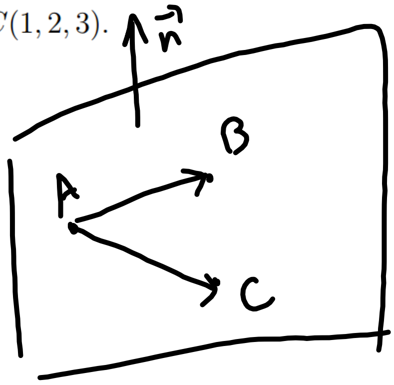
$(6, 0, 0)$

the y-intercept $(0, 4, 0)$

the z-intercept $(0, 0, 3)$



EXAMPLE 7. *Determine the equation of the plane through the points* $A(1,1,1)$, $B(0,1,0)$ *and* $C(1,2,3)$.



$\vec{n} \perp \overrightarrow{AB}$, $\vec{n} \perp \overrightarrow{AC}$ } Thus, one can choose

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\overrightarrow{AB} = \langle -1, 0, -1 \rangle$$

$$\overrightarrow{AC} = \langle 0, 1, 2 \rangle$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \left\langle \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}, -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \right\rangle$$

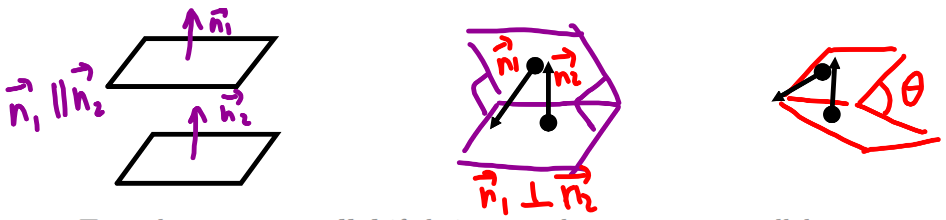
$$= \langle 0+1, -(-2-0), -1-0 \rangle = \langle 1, 2, -1 \rangle$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$1(x-1) + 2(y-1) + (-1)(z-1) = 0$$

$$x - 1 + 2y - 2 - z + 1 = 0$$

$$\boxed{x + 2y - z = 2}$$



$\measuredangle \vec{n}_1, \vec{n}_2 = \theta$

Two planes are **parallel** if their normal vectors are parallel.

Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 8. *Given four planes:*

$$
\begin{array}{lrcrcrcrcl}
P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\
P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\
P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\
P_4: & -2x & + & 3y & + & z & + & 11 & = & 0.
\end{array}
$$

$\vec{n}_1 = \langle 2,3,1\rangle$
$\vec{n}_2 = \langle -4,-6,-2\rangle \parallel \langle 2,3,1\rangle$
$\vec{n}_3 = \langle 2,0,-4\rangle \parallel \langle 1,0,-2\rangle$
$\vec{n}_4 = \langle -2,3,1\rangle$

Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.

(a) P_1 *and* P_2

$\vec{n}_1 \parallel \vec{n}_2 \Rightarrow P_1 \parallel P_2 \Rightarrow \measuredangle P_1, P_2 = 0$

(b) P_1 *and* P_3

$\vec{n}_1 \cdot \vec{n}_3 = \langle 2,3,1\rangle \cdot \langle 1,0,-2\rangle = 2-2=0$

$\Rightarrow \vec{n}_1 \perp \vec{n}_3 \Rightarrow P_1 \perp P_3 \Rightarrow \measuredangle P_1, P_3 = \frac{\pi}{2}$

(c) P_2 *and* P_3

$\left.\begin{array}{l} P_1 \parallel P_2 \\ P_1 \perp P_3 \end{array}\right\} \Rightarrow P_2 \perp P_3 \Rightarrow \measuredangle P_2, P_3 = \frac{\pi}{2}$

(d) P_1 *and* P_4

$\vec{n}_1 \cdot \vec{n}_4 = \langle 2,3,1\rangle \cdot \langle -2,3,1\rangle = -4+9+1 = 6$

$|\vec{n}_1| = |\vec{n}_4| = \sqrt{4+9+1} = \sqrt{14}$

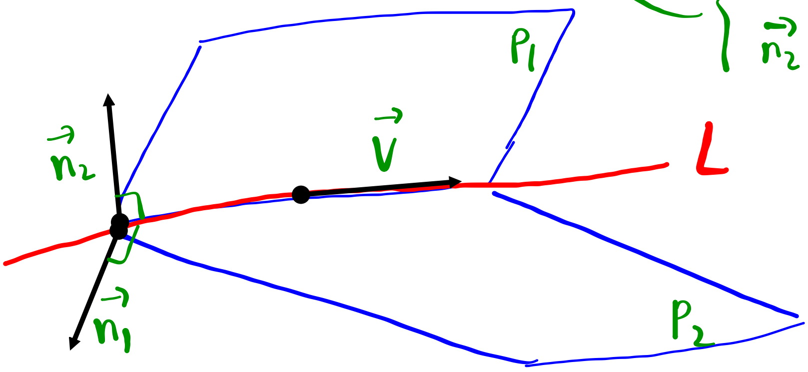
$\cos \measuredangle P_1, P_4 = \cos \measuredangle \vec{n}_1, \vec{n}_4 = \dfrac{\vec{n}_1 \cdot \vec{n}_4}{|\vec{n}_1| \cdot |\vec{n}_4|} = \dfrac{6}{\sqrt{14}\sqrt{14}} = \dfrac{6}{14} = \dfrac{3}{7}$

Line as an intersection of two non parallel planes:

$$L : \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 & P_1 \\ a_2x + b_2y + c_2z + d_2 = 0 & P_2 \end{cases}$$

The direction vector of L is $\vec{v} = \mathbf{n}_1 \times \mathbf{n}_2$.

$$\begin{cases} \vec{n}_1 \perp \vec{v} \\ \vec{n}_2 \perp \vec{v} \end{cases}$$



EXAMPLE 9. *Find an equation of the line given as intersection of two planes:*

$\vec{n}_1 \langle 1, -1, 3 \rangle$

$\vec{n}_2 \langle 1, 1, 4 \rangle$

$$\begin{array}{ccccccc} x & - & y & + & 3z & = & 0 \\ x & + & y & + & 4z & = & 2 \end{array}$$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = \langle -4-3, -(4-3), 1+1 \rangle$$

$$= \langle -7, -1, 2 \rangle$$

$$z = 0 \Rightarrow \begin{cases} x - y = 0 \\ x + y = 2 \end{cases}$$

$$2x = 2 \Rightarrow x = 1 \Rightarrow y = 1$$

$$(1, 1, 0)$$

$$\begin{cases} x = 1 + (-7)t \\ y = 1 + (-1)t \\ z = 0 + 2t \end{cases}$$

$$\boxed{\begin{cases} x = 1 - 7t \\ y = 1 - t \\ z = 2t \end{cases}}$$