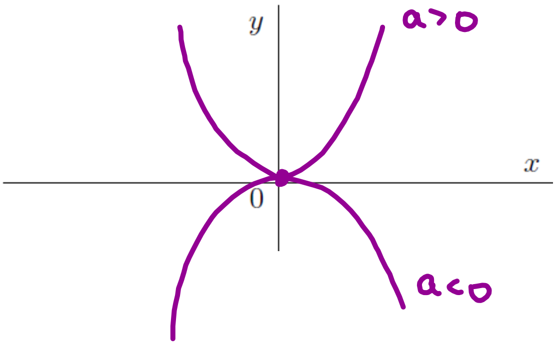
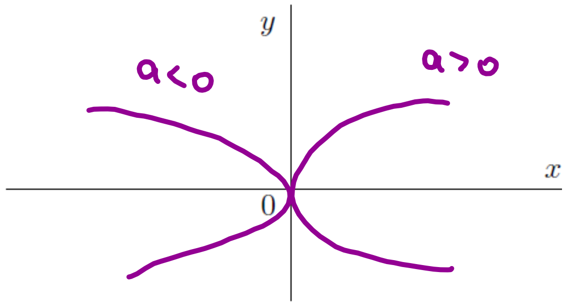
# 11.5: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.

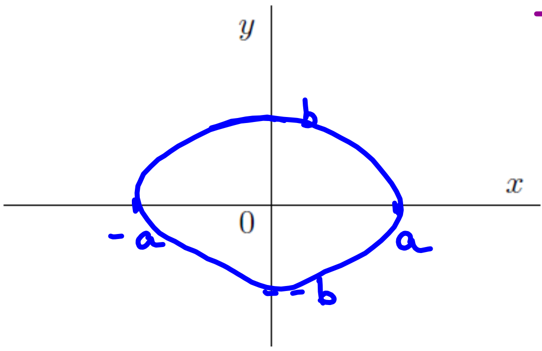
- Parabola: $y = ax^2$ or $x = ay^2$.




- Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

a, b > 0

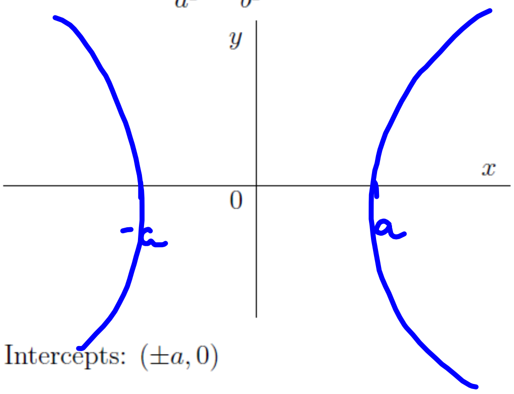



Intercepts
$(\pm a, 0)$
$(0, \pm b)$

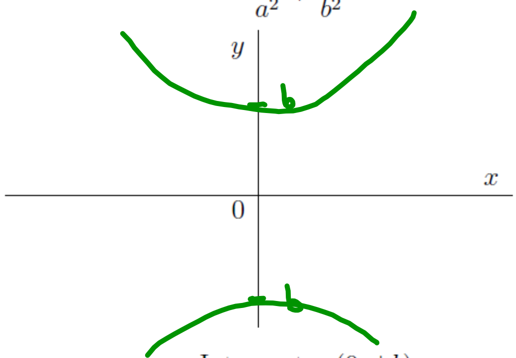
Intercepts: $(\pm a, 0)$ & $(0, \pm b)$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$

- Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$




Intercepts: $(\pm a, 0)$




Intercepts: $(0, \pm b)$

The most general second-degree equation in three variables $x, y$ and $z$:

plane

$$Ax^2 + By^2 + Cz^2 + \underbrace{axy + bxz + cyz} + d_1x + d_2y + d_3z + E = 0, \tag{1}$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

By translations and rotations (1) can be brought into one of the two standard forms:

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0} \quad \text{or} \quad \boxed{Ax^2 + By^2 + Iz = 0.}$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories:

**ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders.** (shown in the table, see Appendix.)

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal ( by planes $z = k$), $yz$-traces (by $x = 0$) and $xz$-traces (by $y = 0$).
3. Intercepts (in some cases).

**To find the equation of a trace** substitute the equation of the plane into the equation of the surface (cf. Example 4, Section 1.1 notes). Note, in the examples

$a, b, c > 0$

below *the constants* $a, b,$ *and* $c$ *are assumed to be positive.*

TECHNIQUES FOR GRAPHING QUADRIC SURFACES

- Ellipsoid. Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Note if $a = b = c$ we have a sphere.

EXAMPLE 1. *Sketch the ellipsoid*

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

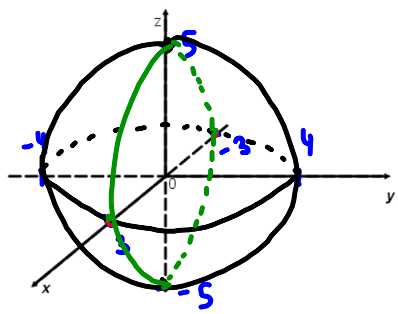
*Solution*

– Find **intercepts**:

$(\pm 3, 0, 0)$ ∗ $x$-intercepts: if $y = z = 0$ then $x = \pm 3$

$(0, \pm 4, 0)$ ∗ $y$-intercepts: if $x = z = 0$ then $y = \pm 4$

$(0, 0, \pm 5)$ ∗ $z$-intercepts: if $x = y = 0$ then $z = \pm 5$




- Obtain traces of:

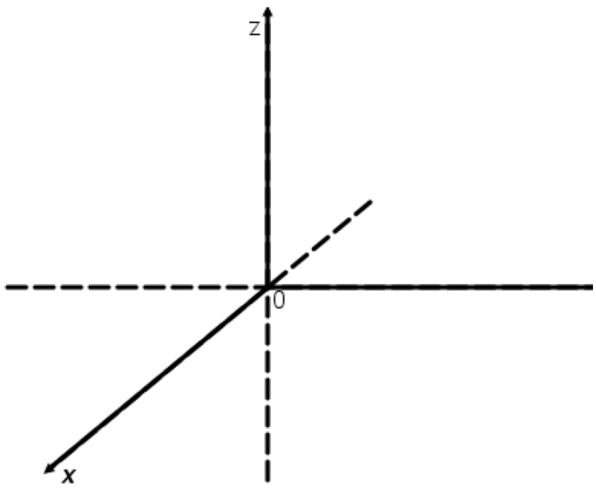
∗ the $xy$-plane: plug in $z = 0$ and get $\frac{x^2}{9} + \frac{y^2}{16} = 1$ ellipse

∗ the $yz$-plane: plug in $x = 0$ and get $\frac{y^2}{16} + \frac{z^2}{25} = 1$

∗ the $xz$-plane: plug in $y = 0$ and get $\frac{x^2}{9} + \frac{z^2}{25} = 1$

- <u>Hyperboloids</u>: There are two types:
  - *Hyperboloid of one sheet.*
    Standard equation:
    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
    EXAMPLE 2. *Sketch the hyperboloid of one sheet*
    $$x^2 + y^2 - \frac{z^2}{9} = 1$$

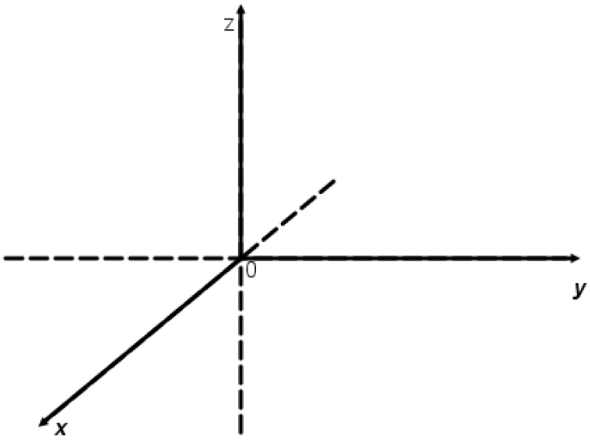
| Plane | Trace |
|---|---|
| $z = 0$ | |
| $z = \pm 3$ | |
| $x = 0$ | |
| $y = 0$ | |



– *Hyperboloid of two sheets.*
Standard equation:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

EXAMPLE 3. *Sketch the hyperboloid of two sheet*




*Solution* Find $z$-ir

| Plane | Trace |
|---|---|
| $z = \pm 2$ | |
| $x = 0$ | |
| $y = 0$ | |

- Elliptic Cones. Standard equation:

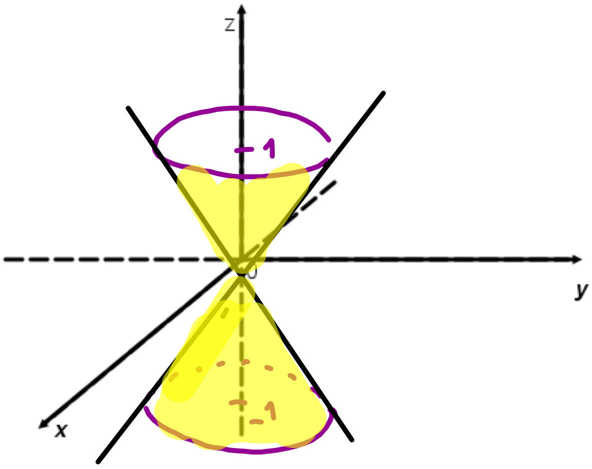
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

If $a=b=c$ then we say that we have a *circular cone.* $x^2+y^2=z^2$

EXAMPLE 4. *Sketch the elliptic cone*

$$z^2=x^2+\frac{y^2}{9}$$

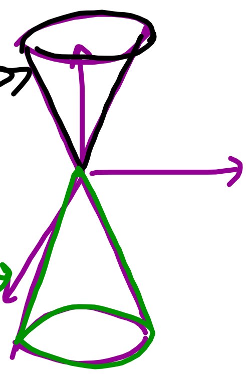
| Plane | Trace |
|---|---|
| $z=\pm 1$ | $x^2+\frac{y^2}{9}=1$ ellipse |
| $x=0$ | $z^2=\frac{y^2}{9} \Rightarrow z=\pm\frac{y}{3}$ two lines through the origin |
| $y=0$ | $z^2=x^2 \Rightarrow z=\pm x$ — ″ — |




Special cases:

$z^2=x^2+y^2$

1. $a=b=c$ circular cone
2. $z=\sqrt{x^2+y^2}$
3. $z=-\sqrt{x^2+y^2}$



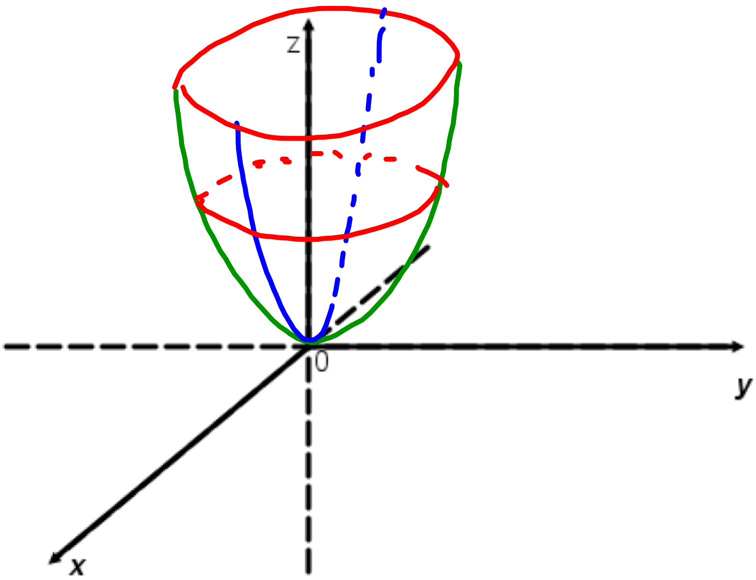
- <u>Paraboloids</u> There are two types:
  - *Elliptic paraboloid.* Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

EXAMPLE 5. *Sketch the elliptic paraboloid*

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

| Plane | Trace |
|---|---|
| $z = 1$ | $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipse |
| $x = 0$ | $z = \frac{y^2}{9}$ parabola |
| $y = 0$ | $z = \frac{x^2}{4}$ — " — |




Special case: $a = b$, $\cancel{c = 1}$

$z = (x^2 + y^2)c$

circular paraboloid

– *Hyperbolic paraboloid.* Standard equation:

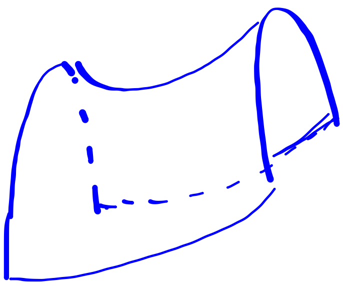
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

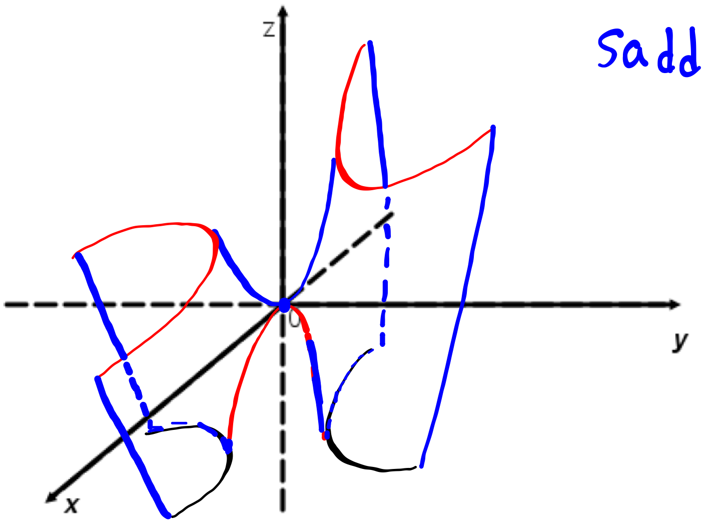
If $z = k$ then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{k}{c}$

EXAMPLE 6. *Sketch the hyperbolic paraboloid*

$$z = x^2 - y^2$$

| Plane | Trace | |
|---|---|---|
| $z = 1$ | $x^2 - y^2 = 1$ | hyperbola |
| $z = -1$ | $x^2 - y^2 = -1$ | |
| $x = 0$ | $z = -y^2$ | parabola |
| $y = 0$ | $z = x^2$ | parabola |






Saddle

- Quadric cylinders: There are three types:

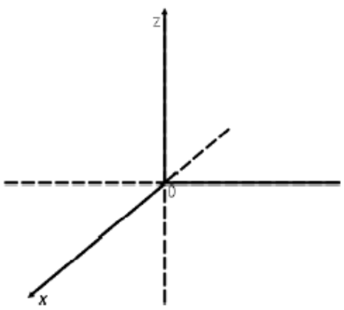
*Elliptic cylinder:*

– Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

EXAMPLE 7. *Sketch elliptic cylinder*

$$x^2 + \frac{y^2}{4} = 1$$



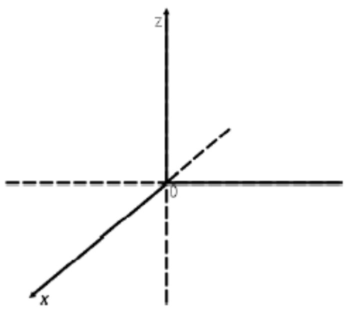

*Hyperbolic cylinder:*

– Standard equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

EXAMPLE 8. *Sketch hyperbolic cylinder*

$$x^2 - y^2 = 1$$



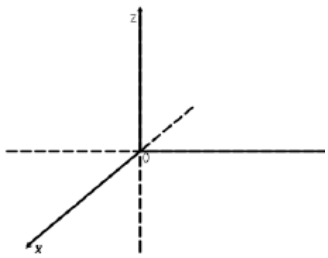

*Parabolic cylinder:*

– Standard equation:

$$y = ax^2$$

EXAMPLE 9. *Sketch parabolic cylinder*

$$y = -x^2$$



CONCLUSION

| | |
|---|---|
| Ellipsoid | $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ |
| Hyperboloid of one sheet | $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ |
| Hyperboloid of two sheets | $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ |
| Elliptic Cones | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ |
| Elliptic paraboloid | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ |
| Hyperbolic paraboloid | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ |
| Elliptic cylinder | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ |
| Hyperbolic cylinder | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ |
| Parabolic cylinder | $y=ax^2$ |

## TRANSLATIONS AND REFLECTIONS OF QUADRIC SURFACES

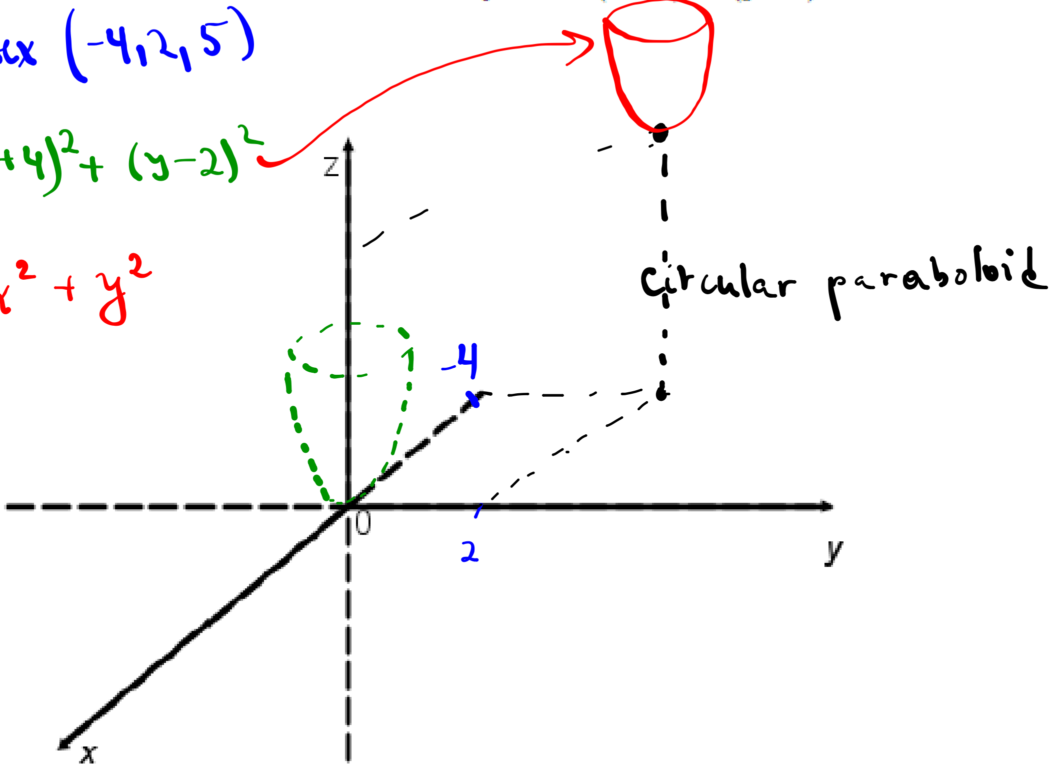
EXAMPLE 10. *Describe and sketch the surface* $z = (x+4)^2 + (y-2)^2 + 5$.

vertex $(-4, 2, 5)$

$z - 5 = (x+4)^2 + (y-2)^2$

vs

$z = x^2 + y^2$



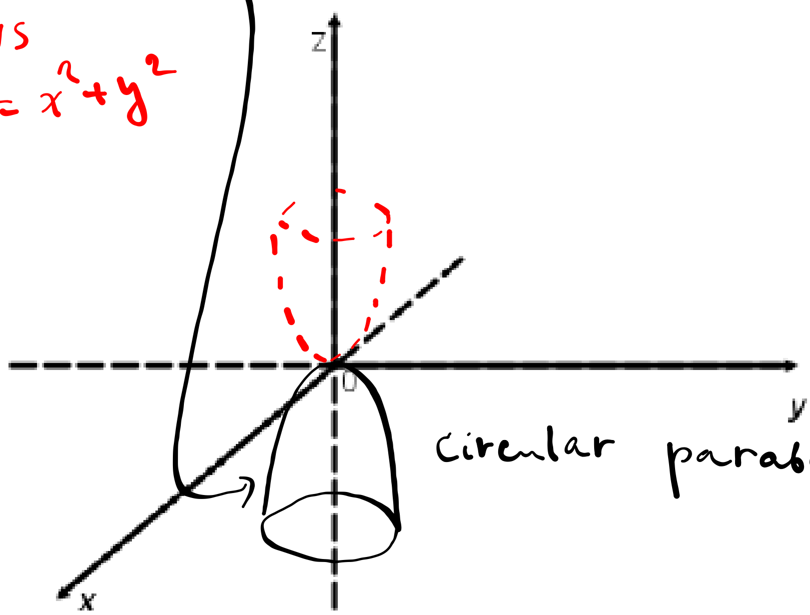
Note that **replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.**

EXAMPLE 11. *Identify and sketch the surface.*

**(a)** $z = -(x^2 + y^2)$

vs

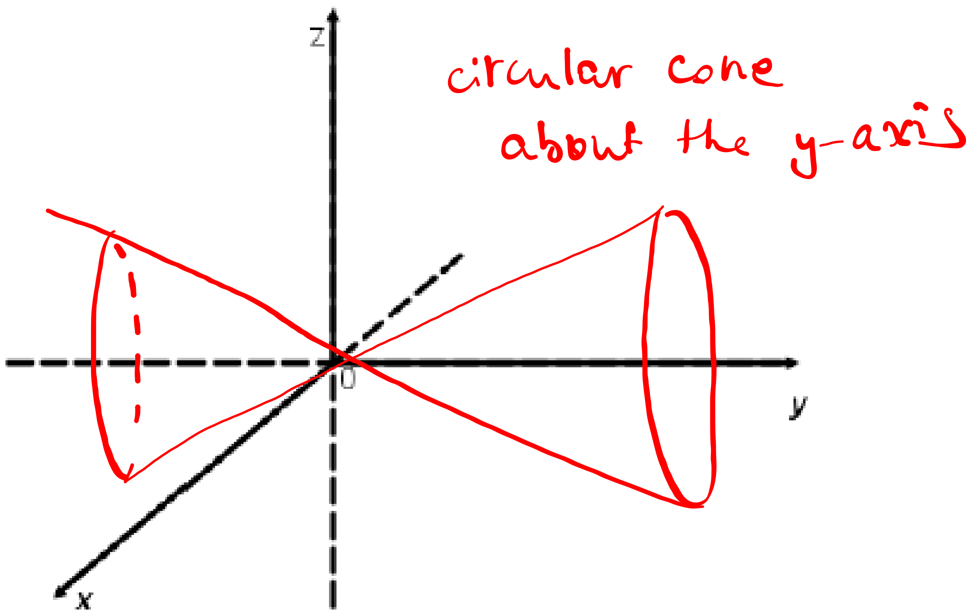
$z = x^2 + y^2$




circular paraboloid

**(b)** $y^2 = x^2 + z^2$

$z^2 = x^2 + y^2$



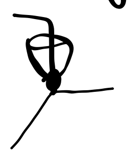
EXAMPLE 12. *Classify and sketch the surface*

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$

Completing squares

$$\left\{\begin{aligned} &x^2 - 4x + 4 \\ &\quad + y^2 - 6y + 9 \\ &\qquad + z = -13 + 4 + 9 \end{aligned}\right.$$

$(x-2)^2 + (y-3)^2 = -z$ vs $z = x^2 + y^2$



+ vertex $(2,3,0)$
+ reflection about the xy-plane